EVOLUTION EQUATIONS AND THEIR APPLICATIONS TO VISCOELASTIC SYSTEMS

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EVOLUTION EQUATIONS AND THEIR APPLICATIONS TO VISCOELASTIC SYSTEMS

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by

Santosh Singh

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CERTIFICATE

It is certified that the work contained in the thesis entitled **"Evolution Equations and Their Applications To Viscoelastic Systems "**, by **Santosh Singh (Roll No: 9610873)**, has been carried out under my supervision and that this work has not been submitted elsewhere for a degree.

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July, 2001.

DEDICATED

TO

MY PARENTS

AND

MY SISTER

Synopsis

Name of the Student:	**Santosh Singh**
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Various problems in physics, for instance, wave propagation and heat conduction in materials with memory effects are of great interests. Various models in continuum mechanics describe the mechanical behavior of solids. In elasticity, an instantaneous response of material is treated as an adequate approximation. Finite elasticity is a part of elasticity theory and does not put any *a priori* limitations on displacement and strains. Such physical problems can be modeled as initial boundary value problems for partial integrodifferential equations. Our main concern is viscoelastic models, where viscoelastic theory may be considered as a theory taking into account of the effect of the entire deformation history on stresses at current instant. In general materials exhibit two different kinds of behaviors, liquid like and solid like. Here we consider the conservation of momentum and energy in viscoelastic media with finite deformations. Such initial boundary value problems may be reformulated as abstract integrodifferential equations in abstract spaces such as Hilbert spaces or Banach spaces. The main advantage of studying these problems in abstract spaces is that we may only have to concentrate on invariant properties of the problems and

need not worry about unnecessary details of a specific problem, and the established results can be applied for the whole class of problems.

We consider the abstract formulations of various physical problems of viscoelastic solid models with short and long memory and related problems. For a material with memory, the stress at an instant of time depends in some fashion on the history of the strain up to the instant of time. The governing equation of motion of a material with memory can be represented as an integrodifferential equations formulated in an abstract space, such as a Hilbert space or a Banach space. By considering their abstract formulations we may concentrate more on the main properties which remain invariant for the whole class. The rich theory of functional analysis may be used to establish wellposedness and the convergence of various type of solutions to the problems under consideration. Our main tools from the functional analysis are the semigroup of operators in a Banach space, theory of monotone (accretive) operators in a Banach space and the method of semidiscritization in time. The main importance of the theory of semigroup appears when we handle the problem with unbounded linear operators. This comes from the fact that when we consider the applications to partial differential equations then the operator associated with such equations are differential operators which are unbounded operators.

There are many types of solutions of a differential or an integrodifferential equation in an abstract space. A form of solution which assumes the existence of derivatives of the solution up to the order of the equation is called **classical solution**. If we relax the condition of the existence of the derivatives upto lesser order, we get other type of weaker form of solutions.

For establishing the existence and uniqueness of strong solutions in the case of short memory we use the method of semidiscritization. Using this method we may not

only establish the existence, uniqueness and regularity of solutions but obtain the approximate solution also in the process. This method so far has been successfully applied to various classes of linear as well as nonlinear parabolic and hyperbolic problems. We replace the time derivatives by their corresponding difference quotients in the time dependent problems. This gives rise to a system of time-independent equations. These systems are guaranteed to have unique solutions, giving us the approximate solutions, by the theory of accretive operators. After establishing some *a priori* estimates for the approximate solutions, one proves the convergence of the approximate solutions to the unique solution of the problem under consideration. The semigroup of operators are obtained as solutions of initial value problems for a first order equation in a Banach space. Most of the theory deals with a single first order equation. The reason for this is that higher order equations can be reduced to first order system and then by changing the underlying Banach space one obtains a single first order equation.

Our aim in the present work is to consider the various physical problems of viscoelastic solid models with short and long memory. As in the case of short memory of viscoelastic materials the problem is reformulated as a second order equation in a Banach space and the method of semi-discretization in time is applied to establish the existence and uniqueness of strong solutions. In case of long memory the problem is reformulated as a single abstract Cauchy equation in a product Hilbert space. In the abstract Cauchy equation the operator is proved to be the infinitesimal generator of contraction semigroup in the product Hilbert space, and then the theory of semigroups is used to obtain the existence, uniqueness and asymptotic behavior of solutions of the problem. We also consider the exponential stability of the semigroup generated by the operator appearing in the abstract Cauchy problem. After obtaining the solutions for viscoelastic system of long and short memory, the regularity of such solutions is established in a Banach space. We also analyzed the

Fadeo-Galerkin approximation of solutions to the integrodifferential equations. We first consider an approximate equations and the existence of a unique solution to this approximate equation. After proving some estimates for the solution of the approximate equation, we establish the convergence of the approximate equation and its solution to the equation under consideration.

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Santosh Singh

Contents

List of Figures

Chapter 1

Literature Survey and Outline of Thesis

1.1 Introduction

The work presented in this thesis deals with the wave propagation and heat conduction in the materials with *memory* effects. The mechanical behavior of solids is described by various models in continuum mechanics. In the present work we mainly concentrate on viscoelastic models. In Elasticity, an instantaneous response of material is treated as an adequate approximation. Finite elasticity is a part of elasticity theory and does not put any *a priori* limitations on displacements and strains. Synthetic rubber, rubber-like polymers and elastomer, biological tissues, metals and alloys under high pressure are some of the examples of elastic media with large deformations.

Viscoelasticity may be considered as a theory taking into account of the effect of the entire deformation history on stresses at current instant. Finite viscoelasticity deals with nonlinear behavior at large deformations. Rubber-like polymers and plastics, metals at elevated temperatures, soils, road construction materials, biological tissues, food-stuffs, polymeric melts and solutions are typical examples of such materials. These materials exhibit two different kinds of behaviors; liquid-like and

solid-like. Here, we are mainly concerned with conservation of momentum and energy in viscoelastic media with finite deformations.

Viscoelastic behavior is typical of a number of materials which are extremely important for applications. We refer to, for instance, Aklonis, MacKnight and Shen [1] for polymers and plastics, Alberola, Lesueur, Granier and Joanicot [2] for composites, Skrzypek [3] for metals and alloys at elevated temperature, Maccarrone and Tiu [4] for road construction materials, Deligianni,Marris and Missirlis [5] for biological tissues, Robert and Sherman [6] for food-stuffs.

1.2 Theory of Viscoelasticity: A brief Introduction

Consider a rod-shaped specimen in its natural stress free state. Suppose we apply a tensile forces P to its ends at instant $t = 0$ and provide the stress $\sigma_0 = P/S$ where S is the cross sectional area. Immediately after application of tensile forces an instant strain ϵ_0 arises in the specimen. In the case of pure elastic material, strain $\epsilon(t)$ at instant $t \geq 0$ coincides with the initial strain ϵ_0. For an inelastic material

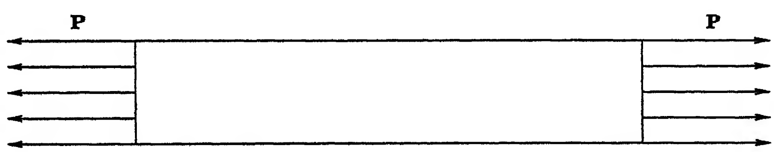

Figure 1.1: A Specimen Under Uniaxial Load

an additional strain $\epsilon(t) > \epsilon_0$ can be observed which increases monotonically with time, i.e., $\dot{\epsilon}(t) \geq 0$, $t \geq 0$, where dot denotes the differentiation with respect to time.

Definition 1.1 This phenomenon of growth of strain under a load uniform in time is called **creep**.

The creep process is characterized either by the creep strain ϵ_c, given by

$$\epsilon_c(t) = \epsilon(t) - \epsilon_0, \tag{1.2.1}$$

or by the creep compliance J which is the ratio of creep strain to the applied stress σ_0, given by

$$J(t) = \frac{\epsilon_c(t)}{\sigma_0} = \frac{\epsilon(t) - \epsilon_0}{\sigma_0}. \tag{1.2.2}$$

The growth of strain in time is a characteristic feature of both inelastic liquids as well as inelastic solids. For solid media, $\epsilon(t) \to \epsilon_\infty$ and $\dot{\epsilon}(t) \to 0$ as $t \to \infty$ with $\epsilon_\infty < \infty$. For liquid media, $\dot{\epsilon}(t) \to \dot{\epsilon}_\infty$ as $t \to \infty$ with $\dot{\epsilon}_\infty < \infty$ corresponding to a Newtonian flow. The Newtonian liquid under stress σ_0 uniform in time, the strain is proportional to time

$$\epsilon_n(t) = \frac{\sigma_0}{\eta} t, \tag{1.2.3}$$

where η is called Newtonian viscosity and $\epsilon_n(t)$ is called Newtonian strain. Taking into account (1.2.1), we may express

$$\epsilon(t) = \epsilon_0 + \tilde{\epsilon}(t) + \epsilon_n(t), \tag{1.2.4}$$

where ϵ_0 is instantaneous elastic strain and $\tilde{\epsilon}_c(t) = \epsilon_c(t) - \epsilon_n(t)$ is the creep proper strain. Equation (1.2.3) is valid for both solid as well as for liquid media with $\eta = \infty$ and $\eta < \infty$, respectively. From (1.2.4), it follows that the creep proper strain tends to a finite limit for solids as well as for liquids. The division of media into solids and liquids is rather conditional as it depends essentially on temperature. Metals at room temperature are elastic solids whereas at elevated temperatures they demonstrate the creep typical of liquids. The strain ϵ_0 characterizes an instant response of material. In applications, media are classified accordingly with instant response, i.e., $\epsilon_0 \neq 0$ and without instant response, i.e., $\epsilon_0 = 0$ (cf. Giesekus [7]).

Consider now another type of loading in which at instant $t = 0$ the specimen is stretched up to a length l_1 and this new length remains fixed for all $t \geq 0$. Then the strain $\epsilon(t) = \epsilon_0$ for all $t \geq 0$. For a purely elastic material the stress σ_0 arises in the specimen at $t = 0$ and remains unchanged during the loading. For an inelastic material the stress $\sigma(t) = \sigma_0$ at $t = 0$ and $\sigma(t)$ decreases monotonically as t increases and tends to a limiting value σ_∞ as $t \to \infty$.

Definition 1.2 The above phenomenon of decreasing of stress in time for a fixed strain is known as **stress relaxation.**

If $\sigma_\infty > 0$, it corresponds to a solid and if $\sigma_\infty = 0$, it corresponds to a liquid. A new shape is created by stretching the specimen. A medium demonstrates the liquid type behavior if no additional stress arises, when a new shape without changes in it's volume is provided. For uniaxial loading, the condition $\sigma_\infty = 0$ means that asymptotically (for large time) the medium accepts its new shape without resistance, i.e. behaves as a liquid.

The stress relaxation modulus $E(t)$ is defined by

$$E(t) = \frac{\sigma(t)}{\epsilon_0}. \tag{1.2.5}$$

For a linear elastic material, its elastic modulus $E = \sigma/\epsilon_0$ is inverse of the compliance $J = \epsilon/\sigma_0$. In general $E(t)$ is not the inverse of $J(t)$. With the notion of creep and relaxation phenomena, we may define a viscoelastic material as follows.

Definition 1.3 A material is called **viscoelastic**, if it exhibits both creep and relaxation phenomena.

Again, this definition is rather conditional. Sometimes we refer to viscoelasticity when only one of these phenomena is taken into account. For example, some applied problems of interest are concerned with either creep or relaxation. Creep and relaxation are processes which correspond to loads uniform in time when either stress or strain remains constant. In applied problems we encounter time-dependent loads. Under these loads the material exhibits a behavior which is not relevant to pure creep or relaxation.

1.3 The Recovery Phenomenon

When any of stress or strain is a step function

$$\sigma(t) = \sigma_o\mathcal{H}(t), \quad \epsilon(t) = \epsilon_o\mathcal{H}(t) \tag{1.3.1}$$

then the creep and relaxation tests are the simplest experiments, where $\mathcal{H}(t)$ is the Heaviside function

$$\mathcal{H}(t) = \begin{cases} 1 & : \quad t \geq 0 \\ 0 & : \quad t < 0 \end{cases} \tag{1.3.2}$$

The above loadings cannot reveal all the characteristic features of the viscoelastic behavior, and more sophisticated programs are necessary. Several tests with time-dependent forces are employed (cf Szczepinski [8]), but the most convenient for applications are experiments with piece-wise constant loads.
The recovery phenomenon is analyzed by using the following form of loading

$$\sigma(t) = \begin{cases} 0 & : \quad t < 0 \\ \sigma_o & : \quad 0 \leq t \leq T \\ 0 & : \quad T < t \end{cases} \tag{1.3.3}$$

The function $\sigma(t)$ and $\epsilon(t)$ corresponding to equation (1.3.3) are plotted as follows. The creep recovery is the difference between the creep strain under loading and the creep strain when external forces and removed. We have two different ways for defining the **creep recovery**.

Definition 1.4 According to Hadley and Ward [9], Ward and Wolfe [10], and Ward [11], the creep recovery equals the difference between strains in two specimen: one of them remains under the action of longitudinal forces and other is unloading : ϵ_r see Fig 1.2 (B)

Definition 1.5 According to Schwartzl [12] and Gramespacher and Meissner [13], the creep recovery equals the difference between the strain at the current moment of time : ϵ_r see Fig 1.2(c)

The second definition is very common for recovery strain. The typical dependencies of the total strain $\epsilon = \epsilon_o + \epsilon_c$ on time t for creep and recovery give rises to three cases:
(i) for small stress, creep and recovery curves coincide;
(ii) for large stresses, the "instantaneous" on short-time recovery is always greater than then "instantaneous" creep.

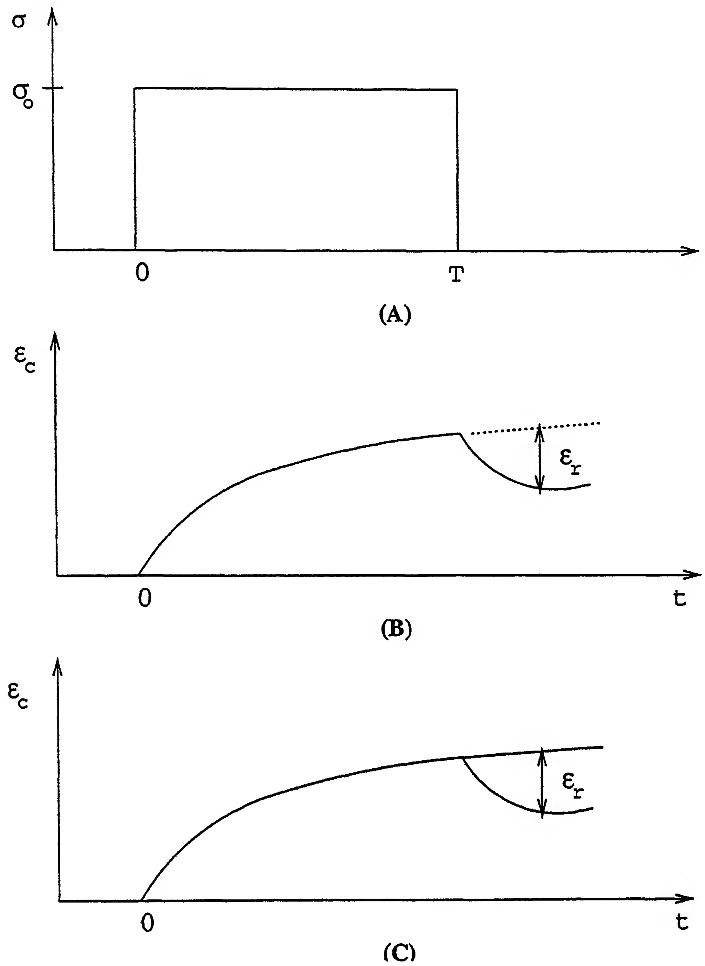

Figure 1.2: A Recovery Program For A Viscoelastic Specimen

(iii) for any level of stresses, the creep and recovery strains increased in time monotonically.

1.4 Nonlinearity of Material Response

A number of elastic materials are rather hard and britle, and they demonstrate the linear response practically in the entire interval of strains before failure. Rubbers

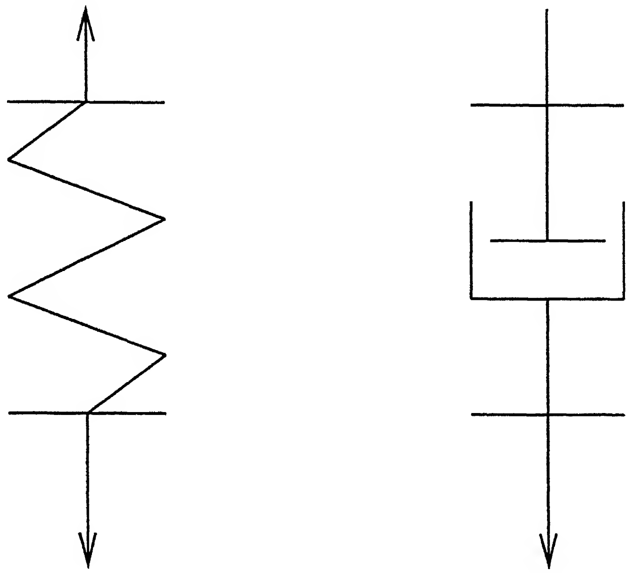

Figure 1.3: Basic Rheological Elements

and biological tissues are the only exceptions which permit large deformations up to several hundred percent.

In viscoelastic media only a few materials demonstrate the linear response, whereas the majority of viscoelastic media are physically nonlinear.

For linear materials, the ratio $\epsilon_c(t)/\sigma_o$ which determines the creep compliance $J(t)$ should be independent of the applied stress σ_o.

1.5 Differential Models with Small Strains

Viscoelastic materials have both elastic and viscous features. In rheology, elastic and viscous properties are described by springs and dash-pots respectively. Fig 1.3 gives the schematic representation of the basic rheological elements.

A linear elastic spring follows the Hooks law

$$\sigma = E\epsilon, \tag{1.5.1}$$

where E is called **Young's modulus**. For a nonlinear elastic string, we have

$$\sigma = \Phi(\epsilon), \tag{1.5.2}$$

where Φ is a smooth function. Viscous properties are modeled by dash-pots. In this case, the linear dependence of stress on the rate of strain is given by the Newton law,

$$\sigma = \eta\dot{\epsilon}, \tag{1.5.3}$$

where η is called **Newtonian viscosity**. Again, in the nonlinear case, we have

$$\sigma = \Psi(\dot{\epsilon}) \tag{1.5.4}$$

with Ψ a smooth function. For basic rheological models, its natural to combine springs and dash-pots in a group. Combining in series leads to Maxwell model and the parallel combination gives rise to Kelvin-Voigt model. For more general rheological models we combine the basic elements (springs and dash-pots) as well as the Maxwell and Kelvin-Voigt elements in parallel and in series.

If we take a Maxwell model with an elastic spring in parallel, we obtain the standard viscoelastic solid known as **Zener model**. The standard viscoelastic solid is one of the basic models for the theoretical analysis as this model describes both creep and relaxation processes and it can be represented both in the differential and integral forms.

We now derive differential equation for a standard viscoelastic solid, for this we use the following rules.
(i)If the elements are connected in parallel, their strains coincide and the total stress is the sum of stresses in separate elements.
(ii) If the elements are connected in series their stresses coincide and the total strain

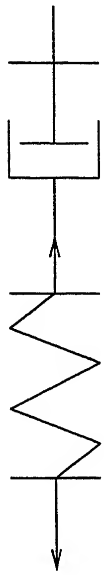

Figure 1.4: The Maxwell Model

equals the sum of strains in separate elements.

Applying these rules to Maxwell model, we have

$$\epsilon = \epsilon_e + \epsilon_v \tag{1.5.5}$$

where ϵ_e and ϵ_v are the strains in the spring and in the dash-pot, respectively. Differentiating (1.5.5), we get

$$\dot{\epsilon} = \dot{\epsilon_e} + \dot{\epsilon_v}. \tag{1.5.6}$$

Assuming the linearity of the elements and using (1.5.1) and (1.5.3), we obtain

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta}. \tag{1.5.7}$$

For Kelvin-Voigt model presented in Fig. 1.3, we have

$$\sigma = \sigma_e + \sigma_v, \tag{1.5.8}$$

where σ_e and σ_v are the stresses in spring and dash-pots, respectively. Assuming the linearity of material response, we get

$$\sigma = E\epsilon + \eta\dot{\epsilon}. \tag{1.5.9}$$

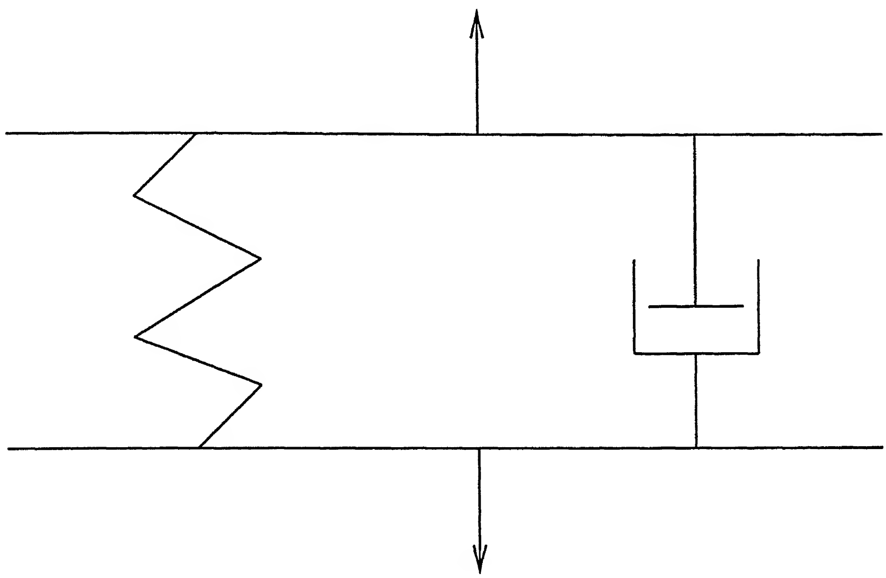

Figure 1.5: The Kelvin-Voigt Model

Now we derive the constitutive equation for standard viscoelastic solid presented in Fig. 1.6. We denote by E and η the Young modulus and the Newtonian viscosity for the Maxwell element and E_1 the Young modulus of the additional spring. Thus, we have

$$\sigma = \sigma_e + \sigma_m, \tag{1.5.10}$$

where σ_e and σ_m are the stresses in the spring and in the maxwell element, respectively. Using (1.5.1) and (1.5.10), we get

$$\sigma_m = \sigma - \sigma_e = \sigma - E_1\epsilon. \tag{1.5.11}$$

From (1.5.11) and (1.5.9), we get the constitutive equation for the viscoelastic solid as

$$\frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} = (1 + \frac{E_1}{E})\dot{\epsilon} + \frac{E_1}{\eta}\epsilon. \tag{1.5.12}$$

For a more general rheological model with an arbitrary number of springs and dashpots, the corresponding constitutive equation takes the form

$$A_0\sigma + A_1\frac{d\sigma}{dt} + \cdots + A_n\frac{d^n\sigma}{dt^n} = B_0\epsilon + B_1\frac{d\epsilon}{dt} + \cdots + B_m\frac{d^m\epsilon}{dt^m}, \tag{1.5.13}$$

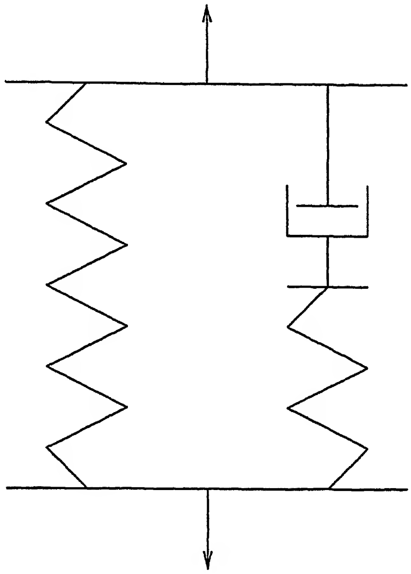

Figure 1.6: The Standard Viscoelastic Solid

where m and n are positive integers and A_k and B_l are material parameters. For $m = n$, Eq. (1.5.13) may be written in integral form. We prove this for the case of standard viscoelastic solid, i.e., $m = n = 1$. In this case we have

$$A_1\dot{\sigma} + A_0\sigma = B_1\dot{\epsilon} + B_0\epsilon. \tag{1.5.14}$$

If $A_1 = 0$, the system is said to have short memory. If $A_1 \neq 0$ the system is said have long memory. In the case of long memory, the constitutive equation (1.5.14) may be rewritten in an integral form. We may rewrite it as first order linear nonhomogeneous equation in σ,

$$\frac{d\sigma}{dt}(t) + \frac{A_0}{A_1}\sigma(t) = \frac{B_1}{A_1}\dot{\epsilon} + \frac{B_0}{A_1}\epsilon. \tag{1.5.15}$$

Using the fact that $\sigma(0) = 0$ and that the linear first order nonhomogeneous equation

$$\frac{dy}{dt}(t) + P(t)y(t) = Q(t), \tag{1.5.16}$$

has the solution

$$y(t) = e^{-\int_{t_0}^{t} P(s)ds}\left[y(t_0) + \int_{t_0}^{t} e^{\int_{t_0}^{s} P(u)du} Q(s)\, ds\right], \tag{1.5.17}$$

where t_0 is any arbitrary constant, we have

$$\sigma(t) = \frac{1}{A_1} e^{-(A_0/A_1)t} \int_0^t \left[B_1 \dot{\epsilon}(s) + B_0 \epsilon(s)\right] e^{(A_0/A_1)s} \, ds. \tag{1.5.18}$$

Using integration by parts and the fact that $\epsilon(0) = 0$, we get

$$\int_0^t \dot{\epsilon}(s) e^{(A_0/A_1)} \, ds = \epsilon(t) e^{(A_0/A_1)t} - \frac{A_0}{A_1} \int_0^t \epsilon(s) e^{(A_0/A_1)s} \, ds. \tag{1.5.19}$$

Using (1.5.19) in (1.5.18), we obtain

$$\sigma(t) = \frac{B_1}{A_1} \epsilon(t) + \frac{B_0}{A_1} \left(1 - \frac{A_0 B_1}{A_1 B_0}\right) \int_0^t \epsilon(s) e^{-(A_0/A_1)(t-s)} \, ds. \tag{1.5.20}$$

Now, the one dimensional equation of motion in the presence of only external force f is given by

$$\rho u_{tt} = \sigma_x + f \tag{1.5.21}$$

where u is the displacement and $\rho > 0$ is the material density. In the case of short memory, i.e., $A_1 = 0$ in the constitutive equation (1.5.14), using the fact that $\epsilon = u_x$, we obtain

$$\rho u_{tt} = C_0 u_{xx} + C_1 u_{xxt} + f \tag{1.5.22}$$

and in the case of long memory we obtain

$$\rho u_{tt}(t, x) = C u_{xx} + D \int_0^t R(t-s) u_{xx}(s, x) \, ds + f(t, x) \tag{1.5.23}$$

for some constants C and D. The initial and boundary conditions may be written as

$$u(0, x) = U_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = U_1(x), \quad x \in (0, l), \tag{1.5.24}$$

$$u(t, 0) = \phi(t), \quad u(t, l) = \psi(t), \quad t \in [0, T]. \tag{1.5.25}$$

Note that the general Dirichlet boundary conditions (1.5.25) can be reduced to homogeneous Dirichlet boundary conditions, i.e., $u(t, 0) = u(t, l) = 0$ for $t \in [0, T]$, by changing $u(t, x)$ to $u(t, x) - (x/l)\psi(t) - ((l - x)/l)\phi(t)$.

1.6 Creep and Relaxation Kernels

We discuss basic properties of creep and relaxation kernels. Let us consider a viscoelastic specimen in its natural (stress-free) state in the form of a rectilinear rod. At instant $\tau \geq 0$, unit tensile forces are applied to the ends of the rod. The longitudinal strain $\rho(t, \tau)$ at moment $t \geq \tau$ is presented as the sum

$$\rho(t, \tau) = \rho_0(\tau) + \rho_1(t, \tau) \tag{1.6.1}$$

where $\rho_0 = E^{-1}(\tau)$ is the instant strain caused; and $e_1 = \rho(t, \tau)$ is an additional (creep) strain caused by the material, $E(\tau)$ is called **current elastic modulus**, and $C(t, \tau)$ is called **creep measure.** Function $C(t, \tau)$ is assumed to be sufficiently smooth and to satisfy the condition

$$C(\tau, \tau) = 0 \tag{1.6.2}$$

Now at the instant $t = 0$ a time varying longitudinal load is applied to the specimen and let $\sigma(t)$ denote the longitudinal stress. Function $\sigma(t)$ is continuously differentiable and satisfies the equality

$$\sigma(0) = 0 \tag{1.6.3}$$

Definition 1.6 The function

$$K(t, \tau) = -E(t)\frac{\partial}{\partial \tau}\left[\frac{1}{E(\tau)} + C(t, \tau)\right] \tag{1.6.4}$$

is called the **creep kernel.**

Now using the creep kernel the constitutive equation for linear viscoelastic media, where as the strain $\epsilon(t)$ at instant t caused by the stress history $\{\sigma(\tau) : 0 \leq \tau \leq t\}$ equals the sum of the strains caused by the elementary strains may be written as

$$\epsilon(t) = \frac{1}{E(t)}\left[\sigma(t) + \int_0^t K(t, \tau)\sigma(\tau)\, d\tau\right] \tag{1.6.5}$$

where as the first term on right hand side determines the instant elastic deformation, while the other term determines the creep deformation.

For a number of materials, the effect of aging is weak and it may be neglected. But

from mathematical point of view, it means the current elastic modulus E can be taken as a constant, $E(t) = E$, while the creep kernel K may be considered as a function of the difference $t-\tau$ only, $K(t,\tau) = K(t-\tau)$. For a non-aging viscoelastic medium, constitutive equation (1.6.5) written as

$$\begin{aligned} \epsilon(t) &= \frac{1}{E}\Big[\sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)\,d\tau\Big] \\ &= \frac{1}{E}\Big[\sigma(t) + \int_0^t \dot{C}_0(t-\tau)\sigma(\tau)\,d\tau\Big] \end{aligned} \tag{1.6.6}$$

where

$$K(t) = \dot{C}_0(t), \quad C_0(t) = EC(t)$$

Equation (1.6.5) and (1.6.6) determine the strain ϵ as a function of the stress σ are called Creep equations. By solving these equations, Relaxation equations are obtained which express the stress σ as a function of the strain ϵ. For the aging viscoelastic medium we get

$$\sigma(t) = E(t)\epsilon(t) - \int_0^t \frac{\partial\chi(t,\tau)}{\partial\tau}\epsilon(\tau)\,d\tau \tag{1.6.7}$$

The function $\chi(t,\tau)$ is presented in the form

$$\chi(t,\tau) = E(\tau) + Q(t,\tau) \tag{1.6.8}$$

where the $Q(t,\tau)$ is called relaxation measure, and satisfy the condition

$$Q(\tau,\tau) = 0 \tag{1.6.9}$$

Definition 1.7 The function

$$R(t,\tau) = \frac{1}{E(t)}\frac{\partial}{\partial\tau}\Big[E(\tau) + Q(t,\tau)\Big] \tag{1.6.10}$$

is called the **relaxation kernel.**

Now Equation (1.6.7) may be rewritten as

$$\sigma(t) = E(t)\Big[\epsilon(t) - \int_0^t R(t,\tau)\epsilon(\tau)\,d\tau\Big] \tag{1.6.11}$$

$$\begin{aligned} \sigma(t) &= E\Big[\epsilon(t) - \int_0^t R(t-\tau)\epsilon(\tau)\,d\tau\Big] \\ &= E\Big[\epsilon(t) - \int_0^t \dot{Q}_0(t-\tau)\epsilon(\tau)\,d\tau\Big] \end{aligned} \tag{1.6.12}$$

from mathematical point of view, it means the current elastic modulus E can be taken as a constant, $E(t) = E$, while the creep kernel K may be considered as a function of the difference $t-\tau$ only, $K(t,\tau) = K(t-\tau)$. For a non-aging viscoelastic medium, constitutive equation (1.6.5) written as

$$\begin{aligned} \epsilon(t) &= \frac{1}{E}\Big[\sigma(t) + \int_0^t K(t-\tau)\sigma(\tau)\,d\tau\Big] \\ &= \frac{1}{E}\Big[\sigma(t) + \int_0^t \dot{C}_0(t-\tau)\sigma(\tau)\,d\tau\Big] \qquad (1.6.6) \end{aligned}$$

where

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where

$$R(t) = -\dot{Q}_0(t), \quad Q_0(t) = \frac{Q(t)}{E} \tag{1.6.13}$$

Definition 1.8 A measure $Q(t,s)$ is called **regular** if it is twice continuously differentiable. If a measure $Q(t,s)$ is only differentiable, and it's derivatives, the relaxation kernel $R(t,s)$, has an integrable singularity at $t = s$, the $Q(t,s)$ is called **weakly singular**.

1.7 Regular and Weakly Singular Kernels

The simplest of regular relaxation measures corresponds to the standard viscoelastic solid and is given as

$$Q_0(t) = -\chi\Big[1 - exp(-\frac{t}{T})\Big] \tag{1.7.1}$$

where χ is the material viscosity, and T is the characteristic time of relaxation. It is assumed that $0 \leq \chi < 1$ and $T > 0$. Using equation (1.6.3) and the differentiating equation(1.6.4) we have

$$R(t) = \frac{\chi}{T}exp\Big(-\frac{t}{T}\Big) \tag{1.7.2}$$

where $R(t)$ is the relaxation kernel.

equation (1.6.4) has important features for applications:

(i) Its simple mechanical interpretation is that, it can be modeled by the system consisting of two elastic springs and dash-post.

(ii) It allows the creep kernel to be found in explicit form.

(iii) Describes qualitatively the material response observed in experiments for both creep and relaxation. In particular, it implies finite positive limits for the creep and relaxation kernels. The power-law relaxation measure, Findly et al. (1989) and Rabotnov (1969)

$$Q_0(t) = -\Big(\frac{t}{T}\Big)^{\alpha} \tag{1.7.3}$$

where $\alpha \in (0,1)$ and $T > 0$ are material parameters. Using equation (1.6.13) and (1.7.3), we have

$$R(t) = \frac{\eta t^{\alpha-1}}{\alpha!} = \eta \mathcal{J}_{\alpha-1}(t) \tag{1.7.4}$$

where $\eta = \alpha(\alpha)!t^{-\alpha}$, $\mathcal{J}_\alpha(t)$ is the Abel Kernel.
In the next chapter, we shall consider the abstract formulations of (1.5.22) and (1.5.23) in a Banach space. These abstract formulations allow us to concentrate on the main features of the problem which are common to a class of such problems. Thus, any result proved for a prototype of problem in a class can be used for all the problems belonging to the class.

1.8 Literature Survey

Our aim in the present work is to consider the abstract formulations of various classes of problems to which (1.5.22), (1.5.23) and related problems are members. By considering their abstract formulations we may concentrate more on the main properties which remain invariant for the whole class. Also, it gives us clear idea of the basic features of a problem and helps in the useful generalization. The rich theory of functional analysis may be used to establish wellposedness and convergence of various type of solutions to the problems under consideration. Our main tools from the functional analysis are the semigroup of operators in a Banach space, theory of monotone (accretive) operators in a Banach space, the method of semi-discretization in time.

Consider, for instance, the differential equation

$$\begin{aligned} \frac{du}{dt} &= Au(t), \quad t > 0 \\ u(0) &= u_0 \end{aligned} \tag{1.8.1}$$

where $u : [0, \infty) \to \mathbf{R}$, and A is a real constant. We know that the solution of (1.8.1) is given by

$$u(t) = u_0 e^{At}, \quad t \geq 0.$$

If we define the maps $T(t) : \mathbf{R} \to \mathbf{R}$ given by

$$T(t)r = re^{At}, \quad t \geq 0,$$

then $T(t)$, $t \geq 0$ is a family of bounded linear operators from $\mathbf{R}$ into $\mathbf{R}$ satisfying the following.

(a) Since $T(0)r = r$, $T(0) = I$, the identity map.
(b) Since

$$\begin{aligned} T(t+s)r &= re^{A(t+s)} = re^{At}e^{As} \\ &= (re^{As})e^{At} = T(t)(re^{As}) \\ &= T(t)T(s)r, \end{aligned} \tag{1.8.2}$$

we have $T(t+s) = T(t)T(s)$ for all nonnegative t and s.
If in (1.8.1) we have $u : [0,\infty) \to \mathbf{R}^n$, $u_0 \in \mathbf{R}^n$ and A is an $n \times n$ matrix, then also the solution has the form

$$u(t) = e^{tA}u_0,$$

where

$$e^{tA} = \sum_{k=0}^{\infty} \frac{t^k A^k}{k!}, \tag{1.8.3}$$

which is majorized by the exponential series

$$\sum_{k=0}^{\infty} \frac{t^k \|A\|^k}{k!}.$$

More generally, if X is a Banach space and $u : [0,\infty) \to X$, $u_0 \in X$ and A is a bounded linear operator from X onto X, the solution of the abstract Cauchy problem,

$$\frac{du}{dt}(t) = Au(t), \quad u(0) = u_0,$$

is given by

$$u(t) = e^{tA}u_0,$$

where e^{tA} is defined similarly as in (1.8.3). Again, in the general situation, if we define for $t \geq 0$, $T(t)x = e^{tA}x$ for every $x \in X$, then $u(t) = T(t)u_0$ satisfies the abstract Cauchy problem. The family $\{T(t),\ t \geq 0\}$ of bounded linear operators in X has the properties (a) and (b). Such a family is called **semigroup of bounded linear operators** in X. The operator A is called the **infinitesimal generator** of the semigroup. Thus if we know the semigroup generated by A then we know the solution of the Cauchy problem. The main importance of the theory of semigroup

appears when we consider A as unbounded linear operator. This comes form the fact that when we consider the applications to partial differential equations then the operator A is associated with a differential operator.

The theory of semigroup is dealt with in many books on functional analysis. One of the books with extensive work on the subject is the classical book by Hille and Phillips [14]. We also refer to other books of Butzer and Berns [15] Dunford and Schwartz [16] Ladas and Lakshmikantham [17], Martin [18], Yosida [19] and Goldstein [20]. After the result of Hille-Yosida theorem in 1948, which gives the characterization of the infinitesimal generator of a *contraction* semigroup, there was a rapid development in the theory of semigroup. By now, it is an extensive mathematical subject with substantial applications to many fields of analysis. Most of the classical results for semigroup of bounded linear operators in a Banach space were generalized to equi-continuous semigroup of class C_0 in locally convex spaces by Yosida [19]. The strongly continuous semigroup are considered in Pazy [21]. For the theory of nonlinear semigroup in Hilbert and Banach spaces, we refer to Barbu [22], Yosida [19].

In the third chapter we consider the case of short memory. We use the method of semi-discretization in time to establish the existence and uniqueness of *strong* solutions. The method of discretization in time was introduced by Rothe [23] and developed by a group working at Prague, see Rektorys [24], Necas [25], Kacur [26], Kartsatos and Zigler [27]. See also, Bahuguna and Raghavendra [28, 29] and Bahuguna, Pani and Raghavendra [30] and references cited therein. This method consists in dividing the time axis and replacing the time derivatives by their corresponding difference quotients in the time dependent problems. This gives rise to a system of time-independent equations. These systems are guaranteed to have unique solutions, giving us the approximate solutions, by the theory of accretive operators. After establishing some *a priori* estimates for the approximate solutions, one proves the convergence of the approximate solutions to the unique solution of the problem under consideration. As mentioned above, semigroup of operators are

obtained as solutions of initial value problems for a first order equation in a Banach space. Most of the theory deals with a single first order equation. The reason for this is that higher order equations can be reduced to first order systems and then by changing the underlying Banach space one obtains a single first order equation. This idea we have used in the fourth chapter. The initial works on linear viscoelastic materials with short as well as long memory is considered by Glowinski, Lions and Tremolieres [31], Duvaut and Lions [32] and others. The further study has been considered by Dafermos [33, 34], Day [35], Desch and Miller [36] - [37], Hannsgen and Wheeler [38, 39], Fabrizo and Lazzari [40], Liu and Zheng [41, 42] and others. We consider a more general problem in the fourth chapter. In the fifth chapter, we consider the regularity of the solutions whose existence and uniqueness is considered in the earlier chapters. In the last two chapters we consider the problem of heat conduction in material with memory and use the idea of fractional powers of a dissipative operators which was developed by Komatsu [43] Kato [44]. This allows us to consider more general nonlinear forcing terms and establish the existence and uniqueness of *classical solutions.*

1.9 Outline of Thesis

In this section we briefly mention the topics covered in the chapters two to seven of the thesis.

Chapter Two: In this chapter we mention the basic notions, definitions and results from the literature. We first give the idea of abstract formulations of various problems and their importance. We then consider the theory of semigroup of bounded linear operators in a Banach space. We then briefly describe the method of semi-discretization in time.

Chapter Three: This chapter deals with the case of short memory of viscoelastic materials. We reformulate the problem as a second order equation in a Banach space and apply the method of semi-discretization in time to establish the existence and

uniqueness of *strong* solutions. We also make some remarks about the extension of results for the existence and uniqueness of *classical solutions*

Chapter Four: This chapter considers the case of long memory. We reformulate the problem as a single abstract Cauchy equation in a product Hilbert space. We prove that the operator appearing in the abstract Cauchy equation is the infinitesimal generator of a contraction semigroup in the product Hilbert space. We then use the theory of semigroup to obtain the existence, uniqueness and asymptotic behavior of solutions of the problem. We also consider the exponential stability of the semigroup generated the operator appearing in the abstract Cauchy problem.

Chapter Five: In this chapter we establish the regularity of the solutions for an abstract integrodifferential equation in a Banach space. The problem considered in this chapter includes the problems considered in the earlier chapters. Thus, the results established in this chapter prove the regularity of solutions obtained in the earlier chapters.

Chapter Six: In this chapter we consider the Fadeo-Galerkin approximation of solutions to the integrodifferential equations analyzed in Chapter Six. We first consider an approximate equation and the existence of a unique solution to this approximate equation. After proving some estimates for the solution of the approximate equation, we establish the convergence of the approximate equation and its solution to the equation under consideration and its solution, respectively.

The relevant references are appended at the end.

Chapter 2

Preliminaries and Basic Results

2.1 Introduction

In this section we mention some of the results required in the ensuing chapters. Most of the results are stated here without proofs but with proper references.

We first consider the idea of an abstract formulation and its importance. We then consider various tools to study such abstract formulations. Some of these tools are the theory of semigroup of bounded linear operators in a Banach space X, monotone (or accretive) operators, method of semi-discretization in time. The notion of a semigroup $S(t)$, $t \geq 0$ of bounded linear operators from X into X and its infinitesimal generator A are closely related with various partial differential equations. The generator A is the partial differential operator appearing in the partial differential equations which model many physical phenomena. A typical example of this operator is

$$A = -\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

which is n-dimensional Laplacian and appears in the heat conduction or reaction-diffusion equations and wave equations.

2.2 Abstract Formulation

In order to explain the basic idea, we consider the following initial boundary value problem for the heat equation:

$$\begin{aligned} \frac{\partial u}{\partial t} - \sum_{i=1}^{n} \frac{\partial^2 u}{\partial t^2{}_i} &= f, \quad \text{on} \quad \Omega \times (0,T), \\ u &= 0 \quad \text{on} \quad \partial\Omega \times [0,T] \\ u(x,0) &= u_0(x) \quad \text{on} \quad \Omega. \end{aligned} \tag{2.2.1}$$

Here Ω is a bounded domain in $\mathbf{R}^n$ with sufficiently smooth boundary $\partial\Omega$. The functions f and u_0 are given and u is to be found.

The generalized problem corresponding to (2.2.1) is obtained by multiplying (2.2.1) by a function v in the space $C_0^\infty(\Omega)$ of infinitely differentiable functions having compact support in Ω and integrating by parts. Here by support of a function $\phi \in C(\Omega)$, we mean the closed set

$$K = \overline{\{x \in \Omega \mid \phi(x) \neq 0\}}$$

We thus obtain

$$\begin{aligned} \frac{d}{dt}\int_\Omega u(x,t)v(x)\,dx &+ \int_\Omega \sum_{i=1}^{n} D_i u(x,t) D_i v(x)\,dx \\ &= \int_\Omega f(x,t)v(x)\,dx, \quad x \in \Omega \\ u(x,0) &= u_0(x), \quad x \in \Omega, \end{aligned} \tag{2.2.2}$$

where $x = (x_1, x_2, \cdots, x_n)$ and $D_i = \partial/\partial x_i$. We consider

$$V = W_0^{1,2}(\Omega), \quad H = L^2(\Omega)$$

where $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$ are the usual Sobolev spaces. (Cf. Pazy [21] for definitions and other details). We observe the following main points,
(a) the treatments to the space variables $x = (x_1, x_2, \cdots, x_n)$ and the time variable t are different. (b) For fixed time t, the function

$$x \longmapsto u(x,t)$$

is an element of the Sobolev space V. We briefly denote this element by $u(t)$, i.e., for each $t \in [0, T]$

$$u(t) \in V.$$

Thus, if we vary t in $[0, T]$ then we obtain a function

$$t \longmapsto u(t).$$

Hence, from the real function $(x, t) \longmapsto u(x, t)$, we obtain a Banach space V valued function $t \longmapsto u(t)$.

We now write the equation (2.2.2) as

$$\begin{aligned} \frac{d}{dt}(u(t), v)_H &+ a(u(t), v) \\ &= (f(t), v)_H \quad \text{on} \quad (0, T), \quad \text{for all} \quad v \in V, \qquad (2.2.3) \\ u(0) &= u_0 \in H. \qquad (2.2.4) \end{aligned}$$

We have to find the function $t \longmapsto u(t) \in V$ for all $t \in [0, T]$. For all w, v, we define

$$\begin{aligned} a(w, v) &= \int_\Omega \sum_{i=1}^{N} d_i w(x) D_i v(x)\, dx, \\ (f(t), v) &= \int_\Omega f(x, t) v(x)\, dx. \end{aligned}$$

Thus we see the natural need of considering two spaces H and V. H is required for the time derivative and V is needed for the space derivatives in the Laplacian

$$-\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

In the present case, we have the embedding $V \hookrightarrow H$ is continuous and V is dense in H. This leads us to the evolution triplet

$$V \subseteq H \subseteq V^*.$$

The time derivative d/dt is considered as the generalized derivative on $(0, T)$. For this reason equation (2.2.3) is required to be satisfied only almost everywhere (written in short as a.e.) on $(0, T)$. For the function $t \longmapsto u(t)$ we choose the Sobolev

space

$$W^{1,2}(0,T;V,H) = \{u \in L^p(0,T;V) \mid u' \in L^q(0,T;V^*)\}.$$

The corresponding norm is given by

$$\|u\| = \left(\int_0^T \|u(t)\|_V^2 \, dt\right)^{1/2} + \left(\int_0^T \|u'(t)\|_{V^*}^2 \, dt\right)^{1/2}$$

where u' is the generalized derivative of u. The generalized derivative of u is defined in such a way that

$$< u'(t), v >_V = \frac{d}{dt}(u(t), v)_H \quad \text{for all} \quad v \in V$$

To write the equation (2.2.3) as operator equation we introduce the operator $A : V \longrightarrow V^*$ and the functional $F(t) \in V^*$ through the equations

$$\begin{aligned} < Aw, v >_V &= a(w, v) \\ < F(t), v >_V &= (f(t), v)_H \end{aligned}$$

for all $v \in V$. We then obtain

$$< u'(t) + Au(t) - F(t), v >_V = 0 \quad \text{on } (0,T), \text{ for all } v \in V$$

which leads to the operator equation

$$\begin{aligned} u'(t) + Au(t) &= F(t), \quad \text{a.e. } t \in (0,T) \\ u(0) &= u_0 \in H. \end{aligned} \tag{2.2.5}$$

The equation (2.2.5) is an abstract first order equation known as first order evolution equation. One point we observe here is that $u(t) \in V$ and $u'(t) \in V^*$ and we necessarily need to consider the evolution triplet $V \subseteq H \subseteq V^*$ in this abstract formulation.

There is still another approach in which we need to consider only one space, namely

$H = L^2(\Omega)$ provided $f(t) \in H$. For this we write the original problem (2.2.1) in the following form

$$u'(t) + A_H u(t) = f(t), \quad \text{for all} \quad t \in (0,T) \tag{2.2.6}$$

$$u(0) = u_0 \tag{2.2.7}$$

where the operator A_H is the H space realization of A:

$$\begin{aligned} D(A_H) &= \{u \in V \mid Au \in H\}, \\ A_H u &= Au \quad \text{for} \quad u \in D(A_H) \end{aligned}$$

2.3 Semigroups and their Generators

Definition 2.1 A one parameter family $S(t)$, $t \geq 0$ of bounded linear operators from X into X is called semigroup of **bounded linear operators** on X if
(i) $S(0) = I$ where I is the identity operator on X,
(ii) $S(t+s) = S(t)S(s)$ for every $t, s \geq 0$.

The linear operator A whose domain is defined by

$$D(A) = \left\{ x \in X \ : \ \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \text{ exists} \right\} \tag{2.3.1}$$

and the action of A on $x \in D(A)$ is given by

$$Ax = \lim_{t \downarrow 0} \frac{S(t)x - x}{t} \tag{2.3.2}$$

is called the **infinitesimal generator** of $S(t)$.
Example Let $S(t) : \mathbf{R} \longrightarrow \mathbf{R}$ be given by

$$S(t) = e^{\lambda t} x_0, \quad t \geq 0, \quad x_0 \in \mathbf{R}.$$

Then $S(t)$ is a semigroup of bounded linear operators on $\mathbf{R}$ and its infinitesimal generator is $A = \lambda I$. ∎

Example Let $S(t) : \mathbf{R}^n \longrightarrow \mathbf{R}^n$, be such that

$$S(t) = e^{tB} x_0, \quad t \geq 0, \quad x_0 \in \mathbf{R}^n$$

where $B = [b_{ij}]$ is an $n \times n$ matrix, b_{ij} are constants. Then $S(t)$ is a semigroup of bounded linear operators on X $\mathbf{R}^n$ and its generator is $A = B$. ∎

Example Let $S(t) : X \longrightarrow X$ be given by

$$S(t) = e^{tB}x_0, \quad t \geq 0, \quad x_0 \in \mathbf{R}$$

where $B : X \longrightarrow X$ is a bounded linear operator. Then $S(t)$ is a semigroup of bounded linear operators on X. and its generator is $A = B$. ∎

Definition 2.2 The semigroup $S(t)$ is called **uniformly continuous** if

$$\lim_{t \downarrow 0} \|S(t) - I\| = 0. \tag{2.3.3}$$

We note that the semigroup $S(t)$ in each of the above examples is uniformly continuous. In fact, we have the following result.

Theorem 2.1 *A linear operator A is the infinitesimal generator of a uniformly continuous semigroup if and only if A is a bounded linear operator. Furthermore, the semigroup $S(t)$ generated by a bounded linear operator $A : X \longrightarrow X$ is unique and*

(a) There exists $\omega \geq 0$ such that $\|S(t)\| \leq e^{\omega t}$.

(b) The map $t \mapsto S(t)$ is differentiable in norm and

$$\frac{dS(t)}{dt} = AS(t) = S(t)A.$$

For applications to partial differential equations, we can not expect the infinitesimal generator of a semigroup to be a bounded linear operator. Therefore, the class of uniformly continuous semigroups is of little use for this purpose. Thus, we need to consider other type of semigroups.

Definition 2.3 A semigroup $S(t), t \geq 0$ of bounded linear operators on X is called **strongly continuous** or $\mathbf{C_0}$ **- semigroup** if

$$\lim_{t \downarrow 0} S(t)x = x$$

for all $x \in X$.

For C_0-semigroups, we have the following main result.

Theorem 2.2 *Let $S(t)$ be a C_0-semigroup and let A be its infinitesimal generator. Then*

(a) There exist constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|S(t)\| \leq Me^{\omega t} \quad \textit{for} \quad t \geq 0.$$

(b) For $x \in X$,

$$\lim_{h \to 0} \frac{1}{h} \int_t^{t+h} S(s)x \, ds = S(t)x.$$

(c) For $x \in X$,

$$\int_0^t S(s)x \, ds \in D(A) \quad \textit{and} \quad A\left(\int_0^t S(s)x \, ds\right) = S(t)x - x.$$

(d) For $x \in D(A)$, $S(t)x \in D(A)$ and

$$\begin{aligned} \frac{d}{dt}S(t)x &= AS(t)x \\ &= S(t)Ax. \end{aligned}$$

(e) For $x \in D(A)$,

$$S(t)x - S(s)x = \int_s^t S(\tau)Ax \, d\tau = \int_s^t AS(\tau) \, d\tau.$$

One of main results in the theory of semigroups of bounded linear operators is the Hille-Yosida theorem which was established in 1948 by Hille [45] and Yosida [19]. This result establishes a characterization of the infinitesimal generator of a semigroup $S(t)$ with the property that

$$\|S(t)\| \leq 1 \quad \text{for} \quad t \geq 0. \tag{2.3.4}$$

Definition 2.4 If a semigroup $S(t)$, $t \geq 0$ of bounded linear operators on X satisfies (2.3.4), it is called C_0-semigroup of contractions on X.

For the characterization of the infinitesimal generator of a semigroup of contractions, we have the following result.

Theorem 2.3 *(Hille-Yosida) A linear operator A is the infinitesimal generator of a C_0-semigroup $S(t)$ of contractions, $t \geq 0$ if and only if*
(i) A is closed and $D(A)$ is dense in X,
(ii) the resolvent set $\rho(A)$ of A contains $[0, \infty)$ and

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}$$

for every $\lambda > 0$.

We note that if $S(t)$ is a C_0-semigroup with A its infinitesimal generator such that for some $\omega \geq 0$,

$$\|S(t)\| \leq e^{\omega t} \tag{2.3.5}$$

then $S_1(t) = e^{-\omega t} S(t)$ is a C_0 semigroup of contractions and its infinitesimal generator is $A - \omega I$. Now we consider the notion of dissipative operators. To define the dissipative operators we require duality map defined from X into the dual space X^* of X, given by

$$F(x) = \{x^* \in X^* : < x^*, x >= \|x\|^2 = \|x^*\|_*^2\} \tag{2.3.6}$$

where $< .,. >$ denotes the duality paring between the dual space X^* of X and X. Clearly, duality map is a multi-valued or set valued map in general. In particular, when X is a Hilbert space then the duality map is single-valued.

Definition 2.5 A linear operator A is called **dissipative** if for every $x \in D(A)$ there exists a x^* such that

$$\text{Re} < x^*, Ax > \leq 0.$$

Following theorem gives the characterization of dissipative operators.

Theorem 2.4 *A linear operator A is dissipative if and only if*

$$\lambda\|x\| \leq \|(\lambda I - A)x\|$$

for all $x \in D(A)$ and $\lambda > 0$.

Another useful characterization of the infinitesimal generator of a C_0-semigroup of contraction is given by the following theorem due to Lumer and Phillips [46].

Theorem 2.5 *(Lumer and Phillips) Let A be a linear operator with dense domain $D(A)$ in X. Then we have the following.*
(a) if A is dissipative and there exists a $\lambda_0 > 0$ such that the range $R(\lambda_0 I - A) = X$, then A is the infinitesimal generator of a C_0-semigroup of contractions on X.
(b) if A is the infinitesimal generator of a C_0-semigroup of contractions on X then $R(\lambda I - A) = X$ for all $\lambda > 0$ and A is dissipative.
Moreover, for every $x \in D(A)$ and every $x^* \in F(X)$, $Re < x^*, Ax > \leq 0$.

As pointed out earlier that what we know is the operator A associated with some partial differential operator and the problem is to find the semigroup generated by it. For a bounded linear operator B, we already know that the semigroup is uniformly continuous and given by e^{tB}. The main task is to find the representation of the semigroup generated by a unbounded linear operator, if there is any. For this purpose we use the complex operator calculus. For a bounded linear operator B we have another representation given by the following theorem.

Theorem 2.6 *Let B be a bounded linear operator. If $\gamma > \|B\|$, then*

$$e^{tB} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{\lambda t}(\lambda I - B)^{-1} d\lambda. \tag{2.3.7}$$

The convergence in (2.3.7) is in the uniform operator topology and uniformly in t on bounded intervals.

The following theorem gives the sufficient conditions for an operator A to be the infinitesimal generator of a C_0-semigroup and a similar representation as in (2.3.7).

Theorem 2.7 *Let A be densely defined operator in X satisfying the following conditions.*
(i) For some $0 < \delta < \pi/2$, $\rho(A) \supset \Sigma_\delta = \{\lambda : |arg\,\lambda| < (\pi/2) + \delta\} \cup \{0\}$.
(ii) There exists a constant M such that

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}$$

for $\lambda \in \Sigma_\delta$ with $\lambda > 0$.

Then, A is the infinitesimal generator of a C_0-semigroup $S(t)$ satisfying $\|S(t)\| \leq C$ for some positive constant C and

$$S(t) \quad = \quad \frac{1}{2\pi i}\int_\Gamma e^{\lambda t}(\lambda I - A)^{-1}d\lambda$$

where Γ is a smooth curve in Σ_δ running from $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ for $\pi/2 < \theta < \pi/2+\delta$. The integral converges for $t > 0$ in the uniform operator topology.

2.3.1 Semigroups of Compact Operators

We recall that a linear operator $T : X \longrightarrow X$ is called **compact operator** if for any bounded subset $D \subset X$, $T(D)$ is relatively compact in X, i.e., the closure $\overline{T(D)}$ of $T(D)$ in X is compact.

Definition 2.6 A C_0-semigroup $S(t)$ on X is called **compact** for $t > t_0$ if for every $t > t_0$, $S(t)$ is a compact operator on X. $S(t)$ is called compact if $S(t)$ is compact for $t > 0$.

Note that we can not allow $S(t)$ to be compact for $t \geq 0$ because in that case the identity operator will be compact and hence the space X has to be finite dimensional. Also, if $S(t_0)$ is a compact operator on X for some $t_0 > 0$, then $S(t) = S(t-t_0)S(t_0)$ for $t > t_0$ and therefore $S(t)$ is compact as $S(t - t_0)$ is bounded. Hence if $S(t)$ is a compact operator for $t \geq t_0$ if and only if $S(t_0)$ is a compact operator. We have the following main result for a compact semigroup.

Theorem 2.8 *Let $S(t)$ be a C_0-semigroup. If $S(t)$ is compact for $t > t_0$, then $S(t)$ is continuous in the uniform operator topology for $t > t_0$.*

The following result gives the characterization of the infinitesimal generator of a compact semigroup.

Theorem 2.9 *Let $S(t)$ be a C_0-semigroup and let A be its infinitesimal generator. $S(t)$ is a compact semigroup if and only if $S(t)$ is continuous in the uniform operator topology for $t > 0$ and $(\lambda I - A)^{-1}$ is compact for $\lambda \in \rho(A)$.*

2.3.2 Analytic Semigroups

This class of semigroups on X extends the notion of a C_0-semigroup $S(t)$ on X where the parameter t lies on the nonnegative real axis to the semigroup $S(z)$ where the parameter z lies in a sector

$$\Delta = \{z : \phi_1 < \arg z < \phi_2\} \tag{2.3.8}$$

which contains the nonnegative real axis and $S(z)$ is analytic on Δ. To be precise, we have the following definition and theorem.

Definition 2.7 Let Δ be as in (2.3.8) and for $z \in \Delta$ $S(z)$ be a bounded linear operator on X. The family $S(z)$, $z \in \Delta$ is an analytic semigroup on X in Δ if
(i) $z \mapsto S(z)$ is analytic in Δ,
(ii) $S(0) = I$ and $\lim_{z\to 0, z\in\Delta} S(z)x = x$ for every $x \in X$,
(iii) $S(z_1 + z_2) = S(z_1)S(z_2)$ for $z_1, z_2 \in \Delta$. A semigroup $S(t)$ is called **analytic** if it is analytic in some sector Δ containing the nonnegative real axis.

We note that the restriction of an analytic semigroup to the nonnegative real axis is a C_0-semigroup. When can we extend a C_0-semigroup to an analytic semigroup in some sector containing nonnegative real axis? From Theorem 2.7, we have that for a densely defined operator A in X satisfying (i) and (ii) of Theorem 2.7 is the infinitesimal generator of uniformly bounded C_0-semigroup $S(t)$. Furthermore, in this case, $S(t)$ can be extended to an analytic semigroup in the sector

$$\Delta_\delta = \{z : |\arg z| < \delta\}$$

and $\|S(z)\|$ is uniformly bounded in every closed subsector $\bar{\Delta}_{\delta'}$, $\delta' < \delta$. We collect all these result in the following theorem.

Theorem 2.10 *Let $T(t)$ be a uniformly bounded C_0-semigroup. Let A be the infinitesimal generator of $S(t)$ and assume that $0 \in \rho(A)$. Then the following are equivalent.*
(a) $S(t)$ can be extended to an analytic semigroup in a sector Δ_δ and $\|S(t)\|$ is uniformly bounded in every closed subsector $\bar{\Delta}_{\delta'}$ of Δ_δ, $\delta' < \delta$.

(b) There exists a constant C such that for every $\sigma > 0$, $\tau \neq 0$,

$$\|((\sigma + i\tau)I - A)^{-1}\| \leq \frac{C}{|\tau|}.$$

(c) There exist $0 < \delta < \pi/2$ and $M > 0$ such that

$$\rho(A) \supset \Sigma = \left\{\lambda : |arg\,\lambda| < \frac{\pi}{2} + \delta\right\} \cup \{0\}$$

and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{|\lambda|}$$

for $\lambda \in \Sigma$, $\lambda \neq 0$.

(d) $S(t)$ is differentiable for $t > 0$ and there exists a constant C such that

$$\|AS(t)\| \leq \frac{C}{t}, \quad t > 0.$$

2.3.3 Fractional Powers of Operators

In the case when $-A$ is the infinitesimal generator of an analytic semigroup, we may define the fractional powers of A. This allows us to consider in the ensuing chapters the nonlinear maps involving differential operators also. As we know from Theorem 2.10 that if A is densely defined closed linear operator satisfying

$$\rho(A)\Sigma^{+} = \{\lambda : 0 < |\arg \lambda| \leq \pi\} \cup V, \tag{2.3.9}$$

where V is a neighborhood of zero, and

$$\|(\lambda I - A)^{-1}\| \leq \frac{M}{1 + |\lambda|}, \quad \lambda \in \Sigma^{+}, \tag{2.3.10}$$

2.4 Method of Semi-discretization in Time

In 1930 Rothe [23] suggested a method in which he converted a one dimensional parabolic initial-boundary value problem into a system of boundary value problems

for ordinary differential equations. He divided the time interval $[0,T]$ into n subintervals $[t_{j-1}^n, t_j^n]$, $j = 1,2,\ldots,n$ where $t_j^n = j.h$ and $h = T/n$, and replaced the partial derivative $\partial u/\partial t$ at (x,t) by the difference quotient

$$\frac{u_j^n(x) - u_{j-1}^n(x)}{h}.$$

This idea was adopted in a large number of later works which treated more general parabolic and hyperbolic problems. For more details on the initial development of the method, we refer to the works of Rektorys [24]. For extension to nonlinear problems, we refer to Nečas [47], Kačur [26], Kartsatos and Zigler [27] and references cited therein.

In these works, estimates have been proved directly for the difference quotients. This requires global Lipschitz condition on the nonlinear forcing terms. Bahuguna et al. [48, 49, 50, 30] have modified the method so that one first proves the *a priori* estimates for the discrete points and then with the help of Local Lipschitz conditions one establishes the *a priori* estimates for the difference quotients. This allows us to consider more general nonlinear forcing terms.

Chapter 3

Viscoelastic Materials With Short Memory

3.1 Introduction

In this chapter we shall consider the case of short memory. We reformulate (1.5.22) as a second order abstract Cauchy problem in a Banach space. Our aim is to study the existence and uniqueness of *strong* solutions and our tool is the method of semi-discretization in time. This method has been successfully applied to various first and second order linear as well as nonlinear unsteady state problems. See, for instance, [28, 29, 30]. The method consists in dividing at each step $n = 1, 2, \cdots$, the finite time axis $[0, T]$ for arbitrary $0 < T < \infty$ into subintervals $[t_{j-1}^n, t_j^n]$, $t_j^n = jh$, $h = T/n$ and replacing the time derivatives

$$\begin{aligned} \frac{\partial u}{\partial t} &= \delta u_j^n = \frac{u_j^n - u_{j-1}^n}{k}, \\ \frac{\partial^2 u}{\partial t^2} &= \delta^2 u_j^n = \frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{k^2} \\ &\cdots. \end{aligned}$$

After establishing *a priori* estimates on these *difference quotients* one finally proves the convergence of the approximate solution

$$U^n(t) = u_{j-1}^n + \frac{1}{h}(t - t_{j-1}^n)(u_j^n - u_{j-1}^n)$$

to the solution of the problem.

3.2 Abstract Formulation of the Problem

In this section we describe the abstract formulation of the problem. We reformulate the problem in an abstract space so that our problem can be treated as one of a large class of such problems. Let $X = L^p(0,l)$, $1 < p < \infty$, and let $\tilde{u} : [0,T] \to X$ be given by

$$\tilde{u}(t)(x) = u(t,x), \quad \text{for all} \quad x \in (0,l), \quad t \in [0,T].$$

Let L be the linear operator defined by

$$\begin{aligned} D(L) &= W^{2,p}(0,l) \cap W_0^{1,p}(0,l), \\ Lu &= -\frac{d^2u}{dx^2} \quad \text{for} \quad u \in D(L), \end{aligned}$$

where $W^{m,p}(\Omega)$ and $W_0^{m,p}(\Omega)$, for an open subset Ω of the n-dimensional Euclidean space $\mathbf{R}^n$, are Sobolev spaces (cf Pazy [21] for the definitions). Under reasonable assumptions on f, ϕ and ψ and homogeneous Dirichlet boundary conditions, Eq. (1.5.22) can equivalently be written in the operator form in X as

$$\begin{aligned} \frac{d^2\tilde{u}(t)}{dt^2} + aL\frac{d\tilde{u}(t)}{dt} + bL\tilde{u}(t) &= \tilde{f}(t), \\ \tilde{u}(0) = U_0, \quad \frac{d\tilde{u}(0)}{dt} &= U_1, \end{aligned} \tag{3.2.1}$$

where $\tilde{f} : [0,T] \to X$ given by

$$\tilde{f}(t)(x) = cf(t,x), \quad \text{for all} \quad x \in (0,l), \quad t \in [0,T]$$

and $a > 0$, b and c are some constants. For notational convenience, we replace $\tilde{u}$ and $\tilde{f}$ by u and f, respectively. Transferring other terms on the right in (3.2.1), we obtain

$$\begin{aligned} \frac{d^2u(t)}{dt^2} + A\frac{du(t)}{dt} + Bu &= f(t) + \alpha u(t) + \beta\frac{du(t)}{dt}, \\ u(0) = U_0, \quad \frac{du(0)}{dt} &= U_1, \end{aligned} \tag{3.2.2}$$

where $A = \lambda_0 I + aL$ for some suitable positive constant λ_0 and for some constants α and β. Here I is the identity operator on X. Also, $D(B) = D(A)$ with

$$Bu = c_1 Au + c_2 u \quad \text{for} \quad u \in D(A)$$

for some constants c_1 and c_2.

We assume the following properties of the linear operators A, B.
(P1) The linear operator A is m-accretive, i.e., for every $u \in D(A)$ and $\lambda > 0$

$$\lambda \|u\| \leq \|\lambda u + Au\|$$

and for every $\lambda > 0$

$$R(I + \lambda A) = X,$$

for $\lambda > 0$ where $R(.)$ is the range of an operator.
(P2) $(I + A)^{-1}$ is compact.
(P3) The linear operator B satisfies

$$\|Bu\| \leq C[\, \|Au\| + \|u\|\,]$$

for $u \in D(B)$, where C is a positive constant.
If the operator $-A$ is the generator of a C_0-semigroup of contractions in X then it follows from Lummer-Phillips theorem (see 1.4.3 in [21]) that A satisfies (P1). The property (P2) follows from the compact embedding of $W_0^{1,p}(0,l)$ in $L^p(0,l)$. Property (P3) is straightforward to check.
By a strong solution to (3.2.2) we mean a function $u : J \longrightarrow X$ such that du/dt is absolutely continuous on J into X, $u(0) = U_0$, $du(0)/dt = U_1$ and (3.2.2) is satisfied a.e. on J.
Our aim is to prove the following main result for the initial data (U_0, U_1) in $D(A) \times D(A)$.

Theorem 3.1 *There exists a unique strong solution to (3.2.2) on J for every pair (U_0, U_1) in $D(A) \times D(A)$.*

3.3 Proof of Theorem 3.1

In this section, we prove Theorem 3.1 with the help of Lemmas 3.3 to 3.6, stated and proved below. For a positive integer n, we consider the discretization

$$[t_{j-1}^n, t_j^n], \quad t_j^n = j.k, \quad k = \frac{T}{n}, \; j = 1, 2, \cdots, n;$$

of the interval J. We call u^n an approximate solution and set

$$\begin{aligned} u_0^n &= U_0, \\ u_{-1}^n &= U_0 - kU_1, \\ u_{-2}^n &= k^2[f(0, U_0, U_1) + AU_0] - 2kU_1 + U_0, \end{aligned} \tag{3.3.1}$$

for every n and discretize (3.2.2) as follows:

$$\delta^2 u_j^n + A\delta u_j^n + Bu_{j-1}^n = f_j^n, \quad j = 1, 2, \cdots, n; \tag{3.3.2}$$

where

$$\begin{aligned} \delta^2 u_j^n &= \frac{u_j^n - 2u_{j-1}^n + u_{j-2}^n}{k^2} \\ \delta u_j^n &= \frac{u_j^n - u_{j-1}^n}{k} \\ f_j^n &= f(t_j^n) + \alpha u_{j-1}^n + \beta \delta u_{j-1}^n, \quad j = 1, 2, \cdots, n. \end{aligned} \tag{3.3.3}$$

The existence and uniqueness of $u_j^n \in D(A)$, $j = 1, \cdots, n$; $n \geq 1$, is a consequence of m-accretivity of A. For notational convenience, we suppress the superscript n and use C to denote a generic positive constant independent of the discretization parameters j, k and n, i.e., we allow C to have different values in the same discussion. We shall use the following lemma which is due to Sloan and Thomme [53] to get the estimates in Lemma 3.3.

Lemma 3.2 *Let $\{w_n\}$ be a sequence of nonnegative real numbers satisfying*

$$w_n \leq \alpha_n + \sum_{h=0}^{n-1} \beta_h w_h, \quad n > 0,$$

where $\{\alpha_n\}$ is a nondecreasing sequence of nonnegative real numbers and $\beta_n \geq 0$. Then

$$w_n \leq \alpha_n \exp\left(\sum_{h=0}^{n-1} \beta_h,\right), \quad n > 0$$

The following lemma gives the uniform bounds for the difference quotients $\{\delta^2 u_j\}$ which in turn imply the uniform bounds of $\{\delta u_j\}$. This is essential for the convergence of the approximate solutions to the solution of (3.2.2).

Lemma 3.3 *For all n and $j = 1, 2, \cdots, n$*

$$\|\delta^2 u_j\| \;\le\; C.$$

Proof. We recall here that by C we shall denote a generic positive constant independent of j, k and n. We allow C to have different values depending only on the constants T, $\|AU_0\|$, $\|AU_1\|$, $\|BU_0\|$, $\|f(0, U_0, U_1)\|$, etc. Therefore, the constants CT, Ce^{CT}, etc., will again be replaced by C.
For $j = 1$ in (3.3.2), we have

$$\delta^2 u_1 + kA\delta^2 u_1 \;=\; f_1 - BU_0 - AU_1.$$

The accretivity of A implies that

$$\|\delta^2 u_1\| \;\le\; C.$$

Now, for $2 \le j \le n$ in (3.3.2) we have

$$\delta^2 u_j + kA\delta^2 u_j \;=\; \delta^2 u_{j-1} - kB\delta u_{j-1} + f_j - f_{j-1}.$$

Again, the accretivity of A implies

$$\|\delta^2 u_j\| \;\le\; \|\delta^2 u_{j-1}\| + k\|B\delta u_{j-1}\| + \|f_j - f_{j-1}\|.$$

Reiterating the above inequality and using (P3), we get

$$\|\delta^2 u_j\| \;\le\; C + Ck\sum_{i=1}^{j-1}\|\delta^2 u_j\| + Ck\sum_{i=1}^{j-1}\|A\delta u_j\|. \tag{3.3.4}$$

Now from (3.3.2), we have

$$\|A\delta u_1\| \;\le\; C + \|\delta^2 u_1\|,$$

and for $2 \le i \le n$, we have

$$\|A\delta u_i\| \;\le\; C + \|\delta^2 u_i\| + Ck\sum_{r=1}^{i-1}\|\delta^2 u_r\| + Ck\sum_{r=1}^{i-1}\|A\delta u_r\|.$$

In view of the fact that $jk \leq T$ for $1 \leq j \leq n$, we have the estimates

$$\|A\delta u_i\| \;\leq\; C\left[1 + \max_{0\leq r\leq i}\|\delta^2 u_r\| + k\sum_{r=0}^{i-1}\|A\delta u_r\|\right].$$

Lemma 3.2 implies that

$$\|A\delta u_i\| \;\leq\; C\left[1 + \max_{0\leq r\leq i}\|\delta^2 u_r\|\right], \quad 1 \leq i \leq n. \tag{3.3.5}$$

Using the estimates of (3.3.5) in (3.3.4) we get

$$\max_{0\leq r\leq j}\|\delta^2 u_r\| \;\leq\; C\left[1 + k\sum_{i=0}^{j-1}\max_{0\leq r\leq i}\|\delta^2 u_r\|\right].$$

Again using Lemma 3.2, we get the desired result.

Remark 1 The estimates of Lemma 3.3 imply that for all n and $j = 1, 2, \cdots, n$, $\|\delta u_j\| \leq C$.

Definition 3.1 We introduce sequences of *polygonal functions* $\{U^n\}$ and $\{V^n\}$ from J into X, given by

$$U^n(t) \;=\; u_{j-1} + (t - t_{j-1})\delta u_j, \quad t \in [t_{j-1}, t_j], \quad j = 1, 2, \cdots, n,$$

$$V^n(t) \;=\; \delta u_{j-1} + (t - t_{j-1})\delta^2 u_j, \quad t \in [t_{j-1}, t_j], \quad j = 1, 2, \cdots, n.$$

Further we define the sequences of step functions $\{X^n\}$ and $\{Y^n\}$, from $(-k, T]$ into X, by

$$X^n(t) \;=\; U_0 \quad \text{for} \quad t \in (-k, 0], \quad X^n(t) \;=\; u_j \quad \text{for} \quad t \in [t_{j-1}, t_j],$$

$$Y^n(t) \;=\; U_1 \quad \text{for} \quad t \in (-k, 0], \quad Y^n(t) \;=\; \delta u_j \quad \text{for} \quad t \in [t_{j-1}, t_j]$$

for $j = 1, 2, \cdots, n$.

Remark 2 The sequences $\{U^n(t) - X^n(t)\}$ and $\{V^n(t) - Y^n(t)\}$ converge to zero uniformly on J as $n \longrightarrow \infty$. Moreover, the functions $\{U^n\}$ and $\{V^n\}$ are Lipschitz continuous on J with uniform Lipschitz constants.

Lemma 3.4 *There exist subsequences $\{U^m\}$ of $\{U^n\}$ and $\{V^m\}$ of $\{V^n\}$ and Lipschitz continuous functions u and v such that $U^m \longrightarrow u$ and $V^m \longrightarrow v$ in $C(J, X)$ (with supremum norm) as $m \longrightarrow \infty$. Furthermore,*

$$\frac{du}{dt} = v \quad on \quad J \quad and \quad \frac{d^2u}{dt^2} = \frac{dv}{dt} \quad a.e. \ on \quad J.$$

Proof. The uniform bounds on $\delta^2 u_j$'s in the estimates (3.3.5) imply that $A\delta u_j$'s (and therefore Au_j's) are uniformly bounded for all n and $j = 1, 2, \cdots, n$. We use (P2) to conclude that there exist subsequence $\{X^m\}$ and $\{Y^m\}$ of $\{X^n\}$ and $\{Y^n\}$, respectively, and functions u and v defined from J into X such that

$$X^m(t) \longrightarrow u(t), \quad \text{and} \quad Y^m \longrightarrow v(t)$$

as $m \longrightarrow \infty$.

Now, the uniform Lipschitz continuity of $\{X^m\}$ and $\{Y^m\}$ imply that the families $\{X^m\}$ and $\{Y^m\}$ are continuous and the convergence of $\{X^m\}$ and $\{Y^m\}$ to u and v imply that $\{X^m(t)\}$ and $\{Y^m(t)\}$ are relatively compact for each t in J. The Ascoli-Arzela theorem implies that

$$U^m \longrightarrow u \quad \text{and} \quad V^m \longrightarrow v$$

in $C(J, X)$ as $m \longrightarrow \infty$.

Since $\{U^m\}$ and $\{V^m\}$ are Lipschitz continuous with uniform Lipschitz constants, it follows that u and v are Lipschitz continuous on J. Now, for each $x^* \in X^*$, we have

$$\begin{aligned} < U^m(t), x^* > &= \int_0^t < \frac{dU^m(s)}{dt}, x^* > ds + < U_0, x^* > \\ &= \int_0^t < V^m(s), x^* > ds + < U_0, x^* > . \end{aligned}$$

Passing to the limit $m \longrightarrow \infty$, we obtain

$$< u(t), x^* > = \int_0^t < v(s), x^* > ds + < U_0, x^* > .$$

Therefore $du/dt = v$ on J. Hence,

$$\frac{d^2u}{dt^2} = \frac{dv}{dt} \quad \text{a.e.} \quad on \quad J.$$

Lemma 3.5 *For $t \in J$, $u(t)$, $du/dt \in D(A)$, $AX^m(t) \rightharpoonup Au(t)$, $AY^m(t) \rightharpoonup A(du(t)/dt)$ uniformly on J as $m \longrightarrow \infty$ and $Au(t)$, $A(du(t)/dt)$ are weakly continuous on J. (By $\rightharpoonup$ we denote the weak convergence in X.)*

Proof. Since $X^m(t) \longrightarrow u(t)$ and $Y^m(t) \longrightarrow du(t)/dt$ uniformly on J as $m \longrightarrow \infty$ and $\{AX^m(t)\}$, $\{AY^m(t)\}$ are uniformly bounded, we use Lemma 2.5 due to Kato [54] to assert that for $t \in J$, $u(t)$, $du/dt \in D(A)$ and $AX^m \rightharpoonup Au(t)$, $AY^m(t) \rightharpoonup A(du(t)/dt)$ uniformly on J as $m \longrightarrow \infty$.
To prove the weak continuity of $Au(t)$ and $A(du(t)/dt)$ on J, let $\{t_s\} \subset J$ be such that $t_s \longrightarrow t \in J$ as $s \longrightarrow \infty$. Then $u(t_s) \longrightarrow u(t)$ and $du(t_s)/dt \longrightarrow du(t)/dt$ as $s \longrightarrow \infty$. Since $Au(t_s)$ and $A(du(t_s)/dt)$ are the limit of $\{AX^m(t_s)\}$ and $\{AY^m(t_s)\}$ as $m \longrightarrow \infty$, $\{Au(t_s)\}$ and $\{A(du(t_s)/dt)\}$ are uniformly bounded. We again use Lemma 2.5 in [54] to get that $Au(t_s) \rightharpoonup Au(t)$ and $A(du(t_s)/dt) \rightharpoonup A(du(t)/dt)$ as $s \longrightarrow \infty$.

Lemma 3.6 *$Au(t)$ and $A(du(t)/dt)$ are Bochner integrable on J.*

Proof. We introduce the operator A^m and J^m, defined from X into X, given by

$$\begin{aligned} J^m x &= \left(I + \frac{1}{m}A\right)^{-1} x \\ A^m x &= AJ^m x \quad \text{for} \quad x \in X \end{aligned}$$

Then, for all x and y in X,

$$\begin{aligned} \|J^m x - J^m y\| &\leq \|x - y\| \\ \|A^m x - A^m y\| &\leq 2m\|x - y\| \end{aligned}$$

and for $x \in D(A)$

$$\|A^m x\| \leq \|Ax\| \quad \text{(cf. Kato [54])}.$$

Now,

$$\begin{aligned} \|A^m U^m(t)\| &\leq \|A^m U^m(t) - A^m X^m(t)\| + \|A^m X^m(t)\| \\ &\leq 2m.\frac{C}{m} + C \end{aligned}$$

Hence $\{A^m U^m(t)\}$ is uniformly bounded. Lemma 2.5 of [54] implies that $A^m U^m(t) \rightharpoonup Au(t)$ uniformly on J as $m \longrightarrow \infty$. Now we can imitate the proof of Lemma 4.6 of [54] to get the required result.

Remark 3 Lemma 3.6 also hold good if the operator A is replaced by the operator B.

We can rewrite equation (3.3.2) as

$$\frac{dV^m(t)}{dt} + AY^m(t) + BX^m(t-k) \;=\; f^m(t), \quad \text{a.e.} \quad \text{on} \quad J, \tag{3.3.6}$$

where

$$\begin{aligned} f^m(0) &= f(0) + \alpha U_0 + \beta U_1 \\ f^m(t) &= f(t_j^m) + \alpha u_{j-1}^m + \beta \delta u_{j-1}^m \quad \text{for} \quad t \in (t_{j-1}^m, t_j^m], \quad j = 1, 2, \cdots, m. \end{aligned}$$

Integrating (3.3.6) over $(0,t)$, we get

$$\begin{aligned} V^m(t) &= U_1 - \int_0^t AY^m(s)\, ds - \int_0^t BX^m(s-k)\, ds \\ &+ \int_0^t f^m(s)\, ds. \end{aligned} \tag{3.3.7}$$

Proof of Theorem 3.1 From (3.3.7), for every $x^* \in X^*$, we have

$$\begin{aligned} < V^m(t), x^* > &= < U_1, x^* > - \int_0^t < AY^m(s), x^* > ds \\ &- \int_0^t < BX^m(s-k), x^* > ds \\ &+ \left\langle \int_0^t f^m(s), x^* \right\rangle ds. \end{aligned}$$

Passing through the limit as $m \longrightarrow \infty$, using the bounded convergence theorem and Lemma 3.5, we get

$$\begin{aligned} < v(t), x^* > &= < U_1, x^* > - \int_0^t < Av(s), x^* > ds \\ &- \int_0^t < Bu(s), x^* > ds \\ &+ \left\langle \int_0^t f(s) + \alpha u(s) + \beta v(s), x^* \right\rangle ds. \end{aligned}$$

The integrand on the right are continuous on J for each fixed x^*, hence $< v(t), x^* >$ is continuously differentiable on J. Using Lemma 3.6 and the boundedness of $Au(t)$, $Bu(t)$ and the continuity of f, we deduce that the strong derivative of $v(t)$ exists and

$$\frac{dv(t)}{dt} + Av(t) + Bu(t) = f(t) + \alpha u(t) + \beta v(t), \quad \text{a.e. on} \quad J. \tag{3.3.8}$$

Since we have

$$\begin{aligned} u(0) &= U_0, \quad v(0) = \frac{du}{dt}(0) = U_1 \quad \text{and} \\ v(t) &= \frac{du}{dt}(t), \end{aligned}$$

the equation (3.3.8) implies that $u(t)$ is a strong solution to (3.2.2).

Uniqueness. We may assume without loss of generality (or we can add eI to A for suitable constant e) that the resolvent set $\rho(-A)$ contains the origin, i.e., $-A$ is invertible. It is easy to show that a strong solution to (3.2.2) is a mild solution to (3.2.2). Now we will show the uniqueness of a mild solution to (3.2.2) which will imply the uniqueness of a strong solution to (3.2.2).

Let u_i, $i = 1, 2$, be two mild solutions to (3.2.2). Let $du_i/dt = v_i$ and $-Au_i = x_i$ for $i = 1, 2$. Then

$$\begin{aligned} x_i(t) &= -AU_0 + (T(t) - I)U_1 \\ &+ \int_0^t (T(t-s) - I)G(s, (-A)^{-1}x_i(s), v_i(s))\, ds, \\ v_i(t) &= T(t)U_1 + \int_0^t T(t-s)G(s, (-A)^{-1}x_i(s), v_i(s))\, ds, \quad i = 1, 2. \end{aligned}$$

Hence

$$\begin{aligned} \|x_1(t) &- x_2(t)\| + \|v_1(t) - v_2(t)\| \\ &\le C \int_0^t [\, \|x_1(s) - x_2(s)\| + \|v_1(s) - v_2(s)\|\,]\, ds. \end{aligned}$$

Applying Gronwall's inequality, we get

$$x_1 = x_2 \quad \text{and} \quad v_1 = v_2 \quad \text{on} \quad J.$$

Hence

$$u_1 = (-A)^{-1}x_1 = (-A)^{-1}x_2 = u_2.$$

This completes the proof. The existence and uniqueness result of Theorem 3.1 guarantees that there exists a unique function $u(t,x)$ satisfying initial and boundary conditions (1.5.24) and (1.5.25) such that for a.e. $x \in (0,l)$, the function $t \mapsto u(t,x)$ is continuously differentiable in $(0,T]$, and the function $t \mapsto (du/dt)(t,x)$ is differentiable a.e. on $(0,T)$ and the equation of motion (1.5.22) is satisfied a.e. on $(0,T) \times (0,l)$ for all pairs $(U_0, U_1) \in W^{2,p}(0,l) \cap W_0^{1,p}(0,l)$.

Chapter 4

Viscoelastic Systems With Finite Memory

4.1 Introduction

Consider one dimensional motion of a linear non-aging viscoelastic medium (cf. Drozdov [56] for more details). The medium occupies the half space $x \geq 0$ initially. At time $t = 0$, the boundary surface $x = 0$ starts moving in the x-direction. We assume that the displacement u_0 of the boundary is time independent. The body forces and surface traction vanish.

The displacement vector $\bar{u} = u^i \bar{e}_i$ has only one non-zero component $u^1 = u(t,x)$. The tensor $\hat{\epsilon} = \epsilon^{ij}\bar{e}_i\bar{e}_j$ of infinitesimal strains also has only one non-zero component

$$\epsilon^{11} = \frac{\partial u(t,x)}{\partial x}.$$

Therefore we have

$$\epsilon = \frac{\partial u(t,x)}{\partial x}, \quad e_{11} = \frac{2}{3}\frac{\partial u(t,x)}{\partial x}, \tag{4.1.1}$$

where ϵ is the first invariant of the tensor $\hat{\epsilon}$ and e_{11} is the only non-zero component of the deviator $\hat{e}$ of the strain tensor $\hat{\epsilon}$.

The constitutive equation for the non-aging viscoelastic material is given by

$$\sigma(t) = 3K\epsilon(t) \tag{4.1.2}$$

$$\hat{s}(t) = 2G\left[\hat{e}(t) - \int_0^t R(t-s)\hat{e}(s)ds\right] + 2\eta\,\frac{d\hat{e}(t)}{dt}, \tag{4.1.3}$$

where σ is the first invariant of the stress tensor $\hat{\sigma}$, K and G are volume and shear moduli of elasticity, η is Newtonian viscosity, $R(t) = -Q_0'(t)$ is a relaxation kernel, $Q_0(t)$ is a relaxation measure, here prime denotes the differentiation with respect to time. From (4.1.1) and (4.1.3), we obtain the following non-zero components of the stress tensor $\hat{\sigma}$,

$$\begin{aligned}\sigma^{11} &= (K+\frac{4}{3}G)\frac{\partial u(t,x)}{\partial x} - \frac{4}{3}G\int_0^t R(t-s)\frac{\partial u(s,x)}{\partial x}\,ds \\ &+ \frac{4}{3}\eta\frac{\partial^2 u(t,x)}{\partial t\partial x}, \end{aligned} \tag{4.1.4}$$

$$\sigma^{22} = K\frac{\partial u(t,x)}{\partial x}, \tag{4.1.5}$$

$$\sigma^{33} = K\frac{\partial u(t,x)}{dt}. \tag{4.1.6}$$

The equation of motion in the absence of body forces is given by

$$\rho\frac{\partial^2\bar{u}}{dt^2} = \bar{\nabla}.\hat{\sigma}, \tag{4.1.7}$$

where ρ is the material density, $\bar{\nabla}$ is the Hamilton operator in the initial configuration and the dot stands for the scalar product. From (4.1.6), we obtain the following integro-differential equation for $u(t,x)$:

$$\begin{aligned}\rho\frac{\partial^2 u(t,x)}{\partial t^2} &= (K+\frac{4}{3}G)\frac{\partial^2 u(t,x)}{\partial x^2} + \frac{4\eta}{3}\frac{\partial^3 u(t,x)}{\partial t\partial x^2} \\ &- \frac{4G}{3}\int_0^t R(t-s)\frac{\partial^2 u(s,x)}{\partial x^2}ds.\end{aligned} \tag{4.1.8}$$

The initial and boundary conditions are given by

$$u(0,x) = 0, \quad \frac{\partial u(0,x)}{\partial t} = 0, \quad (x>0), \tag{4.1.9}$$

$$u(t,0) = u_0 \quad (t>0). \tag{4.1.10}$$

4.2 Abstract Formulation

With simple modification of the relaxation measure $Q_0(t)$ we may write the integro-differential equation (4.1.8) as the following abstract integro-differential equation in

a Hilbert space H:

$$u_{tt}(t) + A\left[\alpha u_t(t) + g(0)u(t) + \int_0^\infty g'(s)u(t-s)ds\right] = 0 \qquad (4.2.1)$$

where $\alpha \geq 0$ is a constant and A is a linear positive definite, self-adjoint unbounded operator on H with the domain $D(A)$ contained in another Hilbert space V, where V is densely and continuously embedded in H. On the kernel g we assume the following conditions:

(G1) $g \in C^2(0,\infty) \cap C[0,\infty)$ and $g' \in L^1(0,\infty)$;

(G2) $g(s) > 0,\ g'(s) < 0,\ g'' > 0$ on $(0,\infty)$;

(G3) the equilibrium elastic modulus $g(\infty) > 0$, which is taken as unity without loss of generality;

(G4) there exist positive constants δ, s_1 and C_g such that

$$g''(s) + \delta g'(s) \geq 0, \quad s \in (0,\infty)$$

and

$$g''(s) \leq C_g|g'(s)|, \quad s \in [s_1,\infty).$$

We note that, for instance, the weakly singular kernel of the form

$$g'(s) = -c_1\frac{e^{-c_2 s}}{s^{c_3}}, \quad c_1, c_2 > 0,\ 0 < c_3 < 1, \qquad (4.2.2)$$

satisfies (G1)-(G4).

The extensive treatment of linear viscoelastic materials with short as well as long memory using variational methods is considered by Glowinski, Lions and Tremolieres [31], Duvaut and Lions [32] and others. The particular case of (4.2.1) in which $A = -\Delta$ (Δ is the Laplace operator) and $\alpha = 0$ has been considered by Dafermos [33] subject to the Dirichlet boundary conditions where the history kernel g satisfies (G2), (G3) and

(G1)$'$ $g' \in C^1([0,\infty)),\ g', g'' \in L^1(0,\infty)$,

with g' assumed to be convex and showed that the energy associated with the system,

$$E(t) = \frac{1}{2}\|u(t)\|_V^2 + \left\|\frac{du(t)}{dt}\right\|_H^2 + \int_0^\infty |g'(s)|\ \|u(t) - u(t-s)\|_V^2\ ds,$$

goes to zero asymptotically. The three parts in $E(t)$ are potential, kinetic and memory energy.

Results of Dafermos [33] were improved in Dafermos [34] by dropping the convexity condition on g'. Day [35] has given some explicit rates of decay of the energy for similar problems. Desch and Miller [36] - [37] considered (4.2.1) (with $\alpha = 0$) under the conditions (G1)$'$, (G2) and (G3) and zero initial history. The assumption (G1)$'$ excludes the singular kernels of the type mentioned in (4.2.2). With additional smoothness on the kernel and zero initial history, Hannsgen and Wheeler [38], [39] established exponential decay of the energy of the system. Fabrizo and Lazzari [40] investigated a three dimensional viscoelastic system with memory with the conditions (G1)-(G3) and

$$g''(s) + \delta g'(s) \geq 0, \quad s \in (0, \infty)$$

for some constant $\delta > 0$. Liu and Zheng [41], [42] investigated (4.2.1) without damping, i.e., $\alpha = 0$, under the conditions (G1)-(G4). Our aim is to generalize the results of Liu and Zheng [41],[42] for strongly damped linear viscoelastic systems with memory (i.e., $\alpha > 0$ in (4.2.1)).

Let H be a Hilbert space with the norm $\|\ \|_H$ and the inner product $(.,.)_H$ and let V be another Hilbert space with the norm $|||\ |||_V$ and the inner product $< .,. >_V$ such that the embedding $V \hookrightarrow H$ is dense and continuous. Let V^* denote the dual space of V of all continuous linear functionals on V. Then the triplet (V, H, V^*) satisfies the continuous embedding

$$V \hookrightarrow H \rightarrow V^* \tag{4.2.3}$$

where we have identified H^* with H itself. Consider a symmetric sesquilinear form σ on V satisfying

$$|\sigma(u,v)| \leq \tilde{C}_\sigma |||u|||_V \, |||v|||_V \quad \text{for all} \quad u, v \in V, \tag{4.2.4}$$

$$\sigma(u,u) \geq \omega |||u|||_V^2 \quad \text{for all} \quad u \in V, \tag{4.2.5}$$

where ω and $\tilde{C}_\sigma > 0$ are constants. The form σ gives rise to a bounded linear operator $\tilde{A} : V \longrightarrow V^*$ through the relation

$$\sigma(u,v) = \left\langle \tilde{A}u, v \right\rangle_{V^*,V} \quad \text{for all} \quad u, v \in V, \tag{4.2.6}$$

where $< f, v >_{V^*,V}$ is the duality paring between $f \in V^*$ and $v \in V$. The operator $A : D(A) \longrightarrow H$ given by

$$D(A) = \{u \in V \; : \; \tilde{A}u \in H\} \tag{4.2.7}$$

is called the restriction of $\tilde{A}$ on H and it is a positive definite self-adjoint operator. Also, $V = D(A^{\frac{1}{2}})$ and

$$\sigma(u, v) = \left(A^{\frac{1}{2}}u, A^{\frac{1}{2}}v\right)_H \quad \text{for} \quad u, v \in V. \tag{4.2.8}$$

We assume that this is the kind of operator we have in (4.2.1).
The space V can also be equipped with an inner product

$$(u, v)_V = \sigma(u, v) \quad for \quad u, v \in V \tag{4.2.9}$$

and the inner product $(.,.)_V$ is equivalent to the inner product $< .,. >_V$. Henceforth, on V we shall use the equivalent inner product $(.,.)_V$ and the corresponding norm $||u||_V^2 = (u, u)_V$ only. In place of (4.2.4), we shall use

$$|\sigma(u, v)| \leq C_\sigma ||u||_V \, ||v||_V \quad \text{for all} \quad u, v \in V, \tag{4.2.10}$$

where $C_\sigma > 0$ is another constant.
We set $W = L^2_{g'}(0, \infty; V)$ which is a Hilbert space of all V-valued, square integrable functions defined on the measure space $((0, \infty), V, |g'|ds)$ equipped with the norm

$$||w||_W^2 = \int_0^\infty |g'(s)| \, ||w(s)||_V^2 \, ds. \tag{4.2.11}$$

Let $\mathcal{H} = V \times H \times W$ be the product space which is a Hilbert space equipped with the norm

$$||z||_{\mathcal{H}}^2 = ||u||_V^2 + ||v||_H^2 + ||w||_W^2, \quad \text{for} \quad z = (u, v, w)^T \in \mathcal{H}. \tag{4.2.12}$$

We set

$$D(\mathcal{A}) = \left\{z \in \mathcal{H} \mid \alpha v + u - \int_0^t g'(s)w(s)ds \in D(A); \; v \in V; \; D_s w \in W; \; w(0) = 0\right\}, \tag{4.2.13}$$

$$\mathcal{A}z = \begin{pmatrix} v \\ -A\left(\alpha v + u - \int_0^\infty g'(s)w(s)ds\right) \\ v - D_s w \end{pmatrix}. \tag{4.2.14}$$

Then (4.2.1) can be written as an abstract Cauchy problem in $\mathcal{H}$,

$$\begin{aligned} \frac{dz}{dt} &= \mathcal{A}z, \quad t > 0, \\ z(0) &= z_0, \end{aligned} \tag{4.2.15}$$

with the help of the transformations

$$v = u_t, \quad w(t,s) = u(t) - u(t-s), \quad t, s \in (0, \infty) \tag{4.2.16}$$

and the initial point taken as

$$z_0 = (\phi, , \psi, \xi)^T \in \mathcal{H}. \tag{4.2.17}$$

4.3 Existence and Uniqueness

For the case of without damping, i.e., $\alpha = 0$, it was proved by Liu and Zheng [42] that the operator $\mathcal{A}$ generates a C_0-semigroup of contractions (cf. Pazy [21] for definitions). This result was proved with the help of the following lemma stated and proved in [42].

Lemma 4.1 *Suppose that function g satisfies the conditions (G1)-(G3). Then for any $w \in W$ with $w(0) = 0$ and $D_s w \in W$, we have*

$$\begin{aligned} |sg'(s)| &\to 0, \quad as \quad s \to 0, \\ |g'(s)| \, \|w(s)\|_V^2 &\to 0, \quad as \quad s \to 0, \\ \int_0^\infty g''(s)\|w(s)\|_V^2 \, ds &< \infty, \\ |g'(s)| \, \|w(s)\|_V^2 &\to 0, \quad as \quad s \to \infty. \end{aligned}$$

We prove the following theorem which extends the results of Liu and Zheng [42].

Theorem 4.2 *Let g satisfy (G1)-(G4). Then the linear operator $\mathcal{A}$ given by (4.2.13) and (4.2.14) is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$.*

Proof If we show that $\mathcal{A}$ is dissipative and the range of $\lambda I - \mathcal{A} = \mathcal{H}$ for some $\lambda > 0$, then from Lumer-Phillips Theorem (cf. Pazy [21] on page 14), it will follow

that $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup of contractions on $\mathcal{H}$. For $z \in \mathcal{H}$, we have

$$\begin{aligned}(\mathcal{A}z, z)_{\mathcal{H}} &= (v,u)_V + \left(-A\left(\alpha v + u - \int_0^\infty g'(s)w(s)\,ds\right), v\right)_H \\ &+ (v - D_s w, w)_W. \end{aligned} \tag{4.3.1}$$

We note that

$$\begin{aligned} -\left(A\left(\alpha v + u - \int_0^\infty g'(s)w(s)\,ds\right), v\right)_H &= -\sigma\left(\alpha v + u - \int_0^\infty g'(s)w(s)\,ds, v\right) \\ &= -\alpha\sigma(v,v) - \sigma(u,v) \\ &+ \int_0^\infty g'(s)\sigma(w(s),v)\,ds. \\ &\le -\sigma(u,v) + \int_0^\infty g'(s)\sigma(w(s),v)\,ds \\ &= -(u,v)_V + \int_0^\infty g'(s)\,(w(s),v)_V\,ds. \end{aligned}$$

Also, since $g' < 0$ on $(0,\infty)$, we have

$$\begin{aligned}(v,w)_W &= \int_0^\infty |g'(s)|\,(v,w(s))_V\,ds \\ &= -\int_0^\infty g'(s)\,(v,w(s))_V\,ds. \end{aligned} \tag{4.3.2}$$

Hence, from (4.3.1) and (4.3.2), we have

$$\begin{aligned}(\mathcal{A}z, z)_{\mathcal{H}} &\le -(D_s w, w)_W \\ &= -\int_0^\infty |g'(s)|\,(D_s w(s), w(s))_V\,ds \\ &= \frac{1}{2}\int_0^\infty g'(s)\,\frac{d}{ds}\|w(s)\|_V^2\,ds \\ &= -\frac{1}{2}\int_0^\infty g''(s)\|w(s)\|_V^2\,ds \\ &\le 0. \end{aligned} \tag{4.3.3}$$

Hence the operator $\mathcal{A}$ is dissipative on $\mathcal{H}$. Next we show that the range $R(\mathcal{A}) = \mathcal{H}$. For any $F = (f_1, f_2, f_3)^T \in \mathcal{H}$, we want to find a $z = (u,v,w)^T \in \mathcal{H}$ such that

$Az = F$ which is equivalent to the following system

$$\begin{aligned} v &= f_1, &(4.3.4)\\ -A\left(\alpha v + u - \int_0^\infty g'(s)w(s)\,ds\right) &= f_2, &(4.3.5)\\ v - D_s w &= f_3. &(4.3.6)\end{aligned}$$

From (4.3.4), we have $v = f_1 \in V$ uniquely determined and from (4.3.6), we have

$$w(s) = sf_1 - \int_0^s f_3(\tau)\,d\tau. \tag{4.3.7}$$

We note that the function $\tilde{f}_1 \equiv f_1$ from $(0,\infty)$ into V is in W since

$$\begin{aligned} \|\tilde{f}_1\|_W^2 &= \int_0^\infty |g'(s)|\, \|\tilde{f}_1(s)\|_V^2\,ds \\ &= \|f_1\|_V^2 \left(\int_0^\infty |g'(s)|\,ds\right). \end{aligned}$$

Therefore from (4.3.6) we have that $D_s w = f_1 + f_3 \in W$. Also, we have $w(0) = 0$. Now as in [42], we show that $w \in W$. For any $T, \epsilon > 0$, we use (G4) and Cauchy-Schwartz inequality to get

$$\begin{aligned} \int_\epsilon^T |g'(s)|\, \|w(s)\|_V^2\,ds &\le \frac{1}{\delta}\int_\epsilon^T g''(s)\|w(s)\|_V^2\,ds \\ &= \frac{1}{\delta}\, g'(T)\|w(T)\|_V^2 - \frac{1}{\delta}\, g'(\epsilon)\|w(\epsilon)\|_V^2 \\ &- \frac{2}{\delta}\int_\epsilon^T g'(s)\,\langle w, D_s w\rangle_V\,ds \\ &\le -\frac{1}{\delta}\, g'(\epsilon)\|w(\epsilon)\|_V^2 + \frac{1}{2}\int_\epsilon^T |g'(s)|\|w(s)\|_V^2\,ds \\ &+ \frac{2}{\delta^2}\int_\epsilon^T |g'(s)|\, \|D_s w\|_V^2\,ds \end{aligned} \tag{4.3.8}$$

From (4.3.8), we have

$$\begin{aligned} \int_\epsilon^T |g'(s)|\, \|w(s)\|_V^2\,ds &\le -\frac{2}{\delta}\, g'(\epsilon)\|w(\epsilon)\|_V^2 \\ &+ \frac{4}{\delta^2}\int_\epsilon^T |g'(s)|\, \|D_s w\|_V^2\,ds. \end{aligned} \tag{4.3.9}$$

Taking the limits as $\epsilon \to 0$ and $T \to \infty$ in (4.3.9) using Lemma 4.1, we obtain

$$\begin{aligned} \|w\|_W^2 &\leq \frac{4}{\delta^2}\int_0^\infty |g'(s)|\ \|D_s w\|_V^2\ w\ ds \\ &= \frac{4}{\delta^2}\ \|D_s w\|_W^2 \end{aligned} \tag{4.3.10}$$

which implies that $w \in W$. Now, since $A : D(A) \longrightarrow H$ is onto, let $h \in D(A)$ be such that $Ah = f_2$. Then

$$u = h - \alpha v + \int_0^\infty g'(s)\left[s f_1 - \int_0^s f_3(\tau)\ d\tau\right]\ ds \tag{4.3.11}$$

solves (4.3.5) uniquely. Thus, for any given $F \in \mathcal{H}$, there is a unique $z \in D(\mathcal{A})$ such that $\mathcal{A}z = F$. Now, we show that there exists a positive constant C_0 such that

$$\|z\|_{\mathcal{H}} \leq C_0\|F\|_{\mathcal{H}}. \tag{4.3.12}$$

From (4.3.5), we have

$$-\left(A\left(\alpha v + u - \int_0^\infty g'(s)w(s)\ ds\right), u\right)_H = (f_2, u)_H$$

which is equivalent to

$$-\sigma\Big(\alpha v + u - \int_0^\infty g'(s)w(s)\ ds, u\Big) = (f_2, u)_H$$

and after rearranging the terms leads to

$$-(f_2, u)_H - \alpha\sigma(v, u) + \int_0^\infty g'(s)\sigma(w(s), u)\ ds = \sigma(u, u). \tag{4.3.13}$$

From (4.2.4), (4.2.5) and (4.3.13), we obtain

$$\begin{aligned} \|u\|_V^2 &= \sigma(u, u) \\ &\leq \|f_2\|_H\|u\|_H + \alpha C_\sigma\|v\|_V\|u\|_V \qquad (4.3.14) \\ &+ C_\sigma\int_0^\infty |g'(s)|\ \|w(s)\|_V\|u\|_V\ ds \\ &\leq C_{VH}\|f_2\|_H\|u\|_V + \alpha C_\sigma\|v\|_V\|u\|_V \\ &+ C_\sigma\int_0^\infty |g'(s)|\|w(s)\|_V\|u\|_V\ ds \qquad (4.3.15) \end{aligned}$$

where C_{VH} is a positive constant for the continuous embedding $V \hookrightarrow H$ such that

$$\|u\|_H \le C_{VH}\|u\|_V.$$

We use the inequality $2ab \le a^2 + b^2$ to get

$$\begin{aligned} C_{VH}\|f_2\|_H\|u\|_V &= C_{VH}\left(\sqrt{3}\,\|f_2\|_H\right)\left(\sqrt{\frac{1}{3}}\,\|u\|_V\right) \\ &\le \frac{3C_{VH}^2}{2}\|f_2\|_H^2 + \frac{1}{6}\|u\|_V^2. \end{aligned} \tag{4.3.16}$$

Similarly, we have

$$\alpha C_\sigma\|v\|_V\|u\|_V \le \frac{3\alpha^2 C_\sigma^2}{2}\|v\|_V^2 + \frac{1}{6}\|u\|_V^2 \tag{4.3.17}$$

and

$$\begin{aligned} C_\sigma\|w(s)\|_V\|u\|_V &\le \frac{3C_\sigma^2}{2}\left(\int_0^\infty |g'(s)|ds\right)\|w(s)\|_V^2 \\ &+ \frac{1}{6}\left(\int_0^\infty |g'(s)|ds\right)^{-1}\|u\|_V^2 \end{aligned} \tag{4.3.18}$$

The inequalities (4.3.18), (4.3.10) and the equation (4.3.6) together with the fact that $v = f_1$ imply that

$$\begin{aligned} C_\sigma\int_0^\infty |g'(s)|\,\|w(s)\|_V\|u\|_V\,ds &\le \frac{3C_\sigma^2}{2}\left(\int_0^\infty |g'(s)|\,ds\right)\|w\|_W^2 + \frac{1}{6}\|u\|_V^2 \\ &\le \frac{6C_\sigma^2}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right)\|D_s w\|_W^2 + \frac{1}{6}\|u\|_V^2 \\ &\le \frac{12C_\sigma^2}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right)\left[\|\tilde{f}_1\|_W^2 + \|f_3\|_W^2\right] + \frac{1}{6}\|u\|_V^2 \\ &\le \frac{12C_\sigma^2}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right)\left[\left(\int_0^\infty |g'(s)|\,ds\right)\|f_1\|_V^2 + \|f_3\|_W^2\right] \\ &+ \frac{1}{6}\|u\|_V^2. \end{aligned} \tag{4.3.19}$$

Combining the inequalities (4.3.15), (4.3.16) and (4.3.19), we obtain

$$\begin{aligned} \|u\|_V^2 &\le \left[3\alpha^2 C_\sigma^2 + \frac{24C_\sigma^2}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right)^2\right]\|f_1\|_V^2 \\ &+ 3C_{VH}^2\|f_2\|_H^2 + \frac{24C_\sigma^2}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right)\|f_3\|_W^2. \end{aligned} \tag{4.3.20}$$

Also, we have

$$\|v\|_H^2 = \|f_1\|_H^2 \le C_{VH}^2 \|f_1\|_V^2 \tag{4.3.21}$$

and

$$\begin{aligned} \|w\|_W^2 &\le \frac{4}{\delta^2}\|D_s w\|_W^2 \\ &\le \frac{8}{\delta^2}\left[\left(\int_0^\infty |g'(s)|\,ds\right)\|f_1\|_V^2 + \|f_3\|_W^2\right]. \end{aligned} \tag{4.3.22}$$

From the inequalities (4.3.20)-(4.3.22), we have

$$\begin{aligned} \|z\|_{\mathcal{H}}^2 &\le \left[C_{VH}^2 + 3\alpha^2 C_\sigma^2 + \frac{8}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right) + \frac{24C_\sigma^2}{\delta^2}\left(\int_0^\infty |g'(s)|\,ds\right)^2\right]\|f_1\|_V^2 \\ &+ 3C_{VH}^2\|f_2\|_H^2 + \frac{8}{\delta^2}\left(1+3C_\sigma^2\right)\left(\int_0^\infty |g'(s)|\,ds\right)\|f_3\|_W^2. \end{aligned} \tag{4.3.23}$$

The inequality (4.3.23) implies that there exists a positive constant C_0 independent of z and F such that

$$\|z\|_{\mathcal{H}} \le C_0\|F\|_{\mathcal{H}}. \tag{4.3.24}$$

Thus $\mathcal{A}$ is invertible and $\mathcal{A}^{-1}$ is in the space $B(\mathcal{H})$ of bounded linear operator from $\mathcal{H}$ into itself. Now, from Theorem 5.2-A in Taylor [57] on page 260, it follows that for any $\lambda > \|\mathcal{A}^{-1}\|_{B(\mathcal{H})}$, the operator $\lambda I - \mathcal{A}^{-1}$ is invertible and $(\lambda I - \mathcal{A}^{-1})^{-1}$ is in $B(\mathcal{H})$. Therefore the operator $\lambda I - \mathcal{A}$ is invertible and belongs to $B(\mathcal{H})$ since

$$\lambda I - \mathcal{A} = -\mathcal{A}(I - \lambda\mathcal{A}^{-1}). \tag{4.3.25}$$

The Lumer-Phillips Theorem now gives the required result. This completes the proof of the theorem.

The following result due to Hille [58] (cf. also, Pazy [21] on page 102) for an abstract Cauchy problem of the type (4.2.15) ensures the existence of a unique continuously differentiable solution.

Theorem 4.3 *Let $\mathcal{A}$ be a densely defined linear operator on $\mathcal{H}$ with nonempty resolvent set $\rho(\mathcal{A})$. The abstract Cauchy problem (4.2.15) has a unique solution in $C^1([0,\infty);\mathcal{H})$ for every $z_0 \in D(\mathcal{A})$ if and only if $\mathcal{A}$ is the infinitesimal generator of a C_0-semigroup on $\mathcal{H}$.*

4.4 The Exponential Stability

In the present section we prove that the semigroup $S(t)$ of contractions on $\mathcal{H}$ generated by $\mathcal{A}$ is exponentially stable. This result extends the corresponding result of Liu and Zheng [42].

To establish the exponential stability of the semigroup $S(t)$, we require the following theorem (cf. Theorem 1.3.2 in [42] on page 4).

Theorem 4.4 *Let $S(t)$, $t \geq 0$ be the semigroup of contractions on $\mathcal{H}$ generated by $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$\rho(\mathcal{A}) \supseteq \{i\beta, \beta \in \mathbf{R}\} \equiv i\mathbf{R} \tag{4.4.1}$$

and

$$\limsup_{|\beta|\to\infty} \|(i\beta I - \mathcal{A})^{-1}\|_{B(\mathcal{H})} < \infty. \tag{4.4.2}$$

where $\mathbf{R}$ is the set of reals.

We have the following result.

Theorem 4.5 *The semigroup $S(t)$, $t \geq 0$ of contractions on $\mathcal{H}$ generated by $\mathcal{A}$ is exponentially stable, i.e., there exist positive constants M and γ such that*

$$\|S(t)\| \leq Me^{-\gamma t}, \quad \textit{for all} \quad t > 0.$$

Proof As we have already shown that $\mathcal{A}$ is invertible and $\mathcal{A}^{-1}$ belongs to $B(\mathcal{H})$, the operator $i\beta\mathcal{A}^{-1} - I$ is invertible for $\beta \in \mathbf{R}$ with $|\beta| < \|\mathcal{A}^{-1}\|^{-1}_{B(\mathcal{H})}$. Hence the operator

$$i\beta I - \mathcal{A} = \mathcal{A}(i\beta\mathcal{A}^{-1} - I)$$

is invertible for $|\beta| < \|\mathcal{A}^{-1}\|^{-1}_{B(\mathcal{H})}$ and $\|(i\beta I - \mathcal{A})^{-1}\|_{B(\mathcal{H})}$ is a continuous function of β in the interval $(-\|\mathcal{A}^{-1}\|^{-1}_{B(\mathcal{H})}, \|\mathcal{A}^{-1}\|^{-1}_{B(\mathcal{H})})$.

If

$$\sup\{\|(i\beta I - \mathcal{A})^{-1}\|_{B(\mathcal{H})} \mid |\beta| < \|\mathcal{A}^{-1}\|^{-1}_{B(\mathcal{H})}\} = M < \infty,$$

then by contraction mapping theorem, the operator

$$i\beta I - \mathcal{A} = (i\beta_0 I - \mathcal{A})(I + i(\beta - \beta_0)(i\beta_0 I - \mathcal{A})^{-1})$$

is invertible for $|\beta_0| < \|\mathcal{A}^{-1}\|_{B(\mathcal{H})}^{-1}$ and $|\beta - \beta_0| < (1/M)$. For $|\beta_0|$ very close to $\|\mathcal{A}^{-1}\|_{B(\mathcal{H})}^{-1}$,

$$\{\beta \mid |\beta| < \|\mathcal{A}^{-1}\|_{B(\mathcal{H})}^{-1} + (1/M)\} \subset \rho(\mathcal{A})$$

and $\|(i\beta I - \mathcal{A})^{-1}\|_{B(\mathcal{H})}$ is a continuous function of β in the interval

$$(-\|\mathcal{A}^{-1}\|_{B(\mathcal{H})}^{-1} - (1/M), \|\mathcal{A}^{-1}\|_{B(\mathcal{H})}^{-1} + (1/M)).$$

Therefore if (4.4.1) is not satisfied, then there exists $\omega \in \mathbf{R}$ with $\|\mathcal{A}^{-1}\|_{B(\mathcal{H})}^{-1} \le |\omega| < \infty$ such that $\{i\beta \mid |\beta| < |\omega|\} \subset \rho(\mathcal{A})$ and $\sup\{\|(i\beta I - \mathcal{A})^{-1}\|_{B(\mathcal{H})} \mid |\beta| < |\omega|\} = \infty$. It follows that there exists a sequence $\beta_n \in \mathbf{R}$ with $\beta_n \to \omega$, $|\beta_n| < |\omega|$ and a sequence of complex vector functions $z_n = (u_n, v_n, w_n) \in D(\mathcal{A})$ such that

$$\|z_n\|_{\mathcal{H}}^2 \;=\; \|u_n\|_V^2 + \|v\|_H^2 + \|w\|_W^2 \;=\; 1$$

for all n and

$$(i\beta_n I - \mathcal{A})z_n \to 0 \quad \text{in } \mathcal{H} \text{ as } \quad n \to \infty. \tag{4.4.3}$$

The convergence in (4.4.3) is equivalent to

$$i\beta_n u_n - v_n \to 0, \quad \text{in} \quad V, \tag{4.4.4}$$

$$i\beta_n v_n + A\left(\alpha v_n + u_n - \int_0^\infty g'(s) w_n(s)\, ds\right) \to 0, \quad \text{in} \quad H, \tag{4.4.5}$$

$$i\beta_n w_n - v_n + D_s w_n \to 0, \quad \text{in} \quad W. \tag{4.4.6}$$

As in (4.3.1), taking the inner product of z_n with (4.4.3) in H and collecting the real parts, we obtain

$$\text{Re } (\mathcal{A}z_n, z_n)_{\mathcal{H}} \;=\; -\alpha\|v_n\|_H^2 - \frac{1}{2}\int_0^2 g''(s)\|w_n(s)\|_V^2\, ds \to 0. \tag{4.4.7}$$

From (4.4.7) and (G4), it follows that $v_n \to 0$ in H and $w_n \to 0$ in W. Hence

$$\|u_n\|_V^2 \to 1. \tag{4.4.8}$$

Now, taking the inner product of v_n with (4.4.4) in H, and using the fact that $v_n \to 0$ in H, we get

$$i\beta_n(u_n, v_n)_H \to 0. \tag{4.4.9}$$

Chapter 5

Regularity of Solutions

5.1 Introduction

Consider the following initial value problem for the semi linear hyperbolic integrodifferential equation in the n-dimensional Euclidean space $\mathbf{R}^n$,

$$\begin{aligned}
\frac{\partial^2 u(x,t)}{\partial t^2} &= \Delta u(x,t) + f(t, u(x,t), \frac{\partial u(x,t)}{\partial t}) \\
&+ \int_{t_0}^{t} k(t-s) g(s, u(x,s), \frac{\partial u(x,s)}{\partial s})\, ds, \quad t > t_0,\ x \in \mathbf{R}^n, \quad (5.1.1) \\
u(x,t_0) &= u_1(x),\ \frac{\partial u(x,t_0)}{\partial t} = u_2(x),\ x \in \mathbf{R}^n,
\end{aligned}$$

where Δ is the n-dimensional Laplacian, f and g are smooth nonlinear functions and k is a locally p-integrable function for $1 < p < \infty$. Problem (5.1.1) is equivalent to the first order system

$$\begin{aligned}
\frac{\partial}{\partial t}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} &= \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}\begin{pmatrix} u(x,t) \\ v(x,t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t,u(x,t),v(x,t)) \end{pmatrix} \\
&+ \int_{t_0}^{t} k(t-s)\begin{pmatrix} 0 \\ g(s,u(x,s),v(x,s)) \end{pmatrix} ds, \quad t > t_0,\ x \in \mathbf{R}^n
\end{aligned}$$

and

$$\begin{pmatrix} u(x,t_0) \\ v(x,t_0) \end{pmatrix} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}, \quad x \in \mathbf{R}^n.$$

Thus, for some appropriate choice of a Banach space X of functions, problem (5.1.1) can be treated as a particular case of the following integrodifferential equation in X,

$$\begin{aligned} \frac{du(t)}{dt} + Au(t) &= f(t,u(t)) + \int_{t_0}^{t} k(t-s)g(s,u(s))\,ds, \quad t > t_0, \qquad (5.1.2)\\ u(t_0) &= u_0. \end{aligned}$$

where A, defined from the domain $D(A) \subset X$ into X, is a linear operator, the nonlinear maps f and g are defined from $J_T \times X$ into X, with J_T being the closure of the interval $[t_0, T)$, $t_0 < T \leq \infty$, and the real-valued map k is defined on J_T. For (5.1.1), the right choice of the space is the Hilbert space $H = H^1(\mathbf{R}^n) \times L^2(\mathbf{R}^n)$. The operator $-A$ is associated with the matrix of operators $\begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$ and it is known that this matrix of operators is the infinitesimal generator of a C_0 semigroup of bounded linear operators in H (cf. Pazy [21]).

Initial studies on problem (5.1.2) for the case in which $k \equiv 0$ was considered by Segal [59]. Different forms of solutions in this particular case have been considered by Martin [18] and Pazy [21] . An example in which the mild solution is not a strong solution is given by Webb [60]. Further modifications of results of Segal [59] have been considered by Kato [54], Martin [18] and Pazy [21].

Our aim here is to establish the existence and uniqueness of regular solutions to (5.1.2) which will in turn guarantee the existence and uniqueness of regular solutions to (5.1.1). We first prove the local existence and uniqueness of a mild solution to (5.1.2) under the assumptions that $-A$ generates a C_0 semigroup of bounded linear operators in X, $f(t,u)$ and $g(t,u)$ are continuous in t and satisfy certain local Lipschitz conditions in u with Lipschitz constants depending on t and $\|u\|_X$ and $k \in L^p(0,T)$, $1 < p < \infty$. Under the assumption of continuity on k on $[0,T)$ we show that if the local mild solution exists on $J_{t_1} = [t_0, t_1]$, $t_0 < t_1 < T$, then it can be extended further. Also, we show that either the solution exists on the whole J_T or there is a maximal interval of existence $[t_0, t_{max})$, $t_0 < t_{max} < T$ and

$\lim_{t \uparrow t_{max}} \|u(t)\|_X = \infty$. Furthermore, we show that (5.1.2) has a classical (continuously differentiable) solution provided f and g are continuously differentiable from $[t_0, T) \times X$ into X.

For the various regularity results in the case when the operator $-A$ in (5.1.2) generates an analytic semigroup, we refer to Rankin [61] and Bahuguna [49] and papers cited therein. Note that for the applications to hyperbolic problems, we cannot assume that $-A$ generates an analytic semigroup.

The results presented here can be generalized for the case of time dependent operator $A(t)$ using the theory of evolution systems. For simplicity of presentation, we restrict ourselves to the case $A(t) \equiv A$.

5.2 Main results

Let X be a Banach space and let $\|.\|_X$ denote the norm in X. Let $t_0 \geq 0$ and let $t_0 < T \leq \infty$. Let J_η denote the closure of the interval $[t_0, \eta)$ for any $t_0 < \eta \leq \infty$. Let $C(J_\eta : X)$ be the space of all continuous functions from J_η into X. For a compact interval J_η, let $C(J_\eta : X)$ denote the Banach space of all continuous functions from J_η into X endowed with the supremum norm

$$\|f\|_{C(J_\eta:X)} = \sup_{t \in J_\eta} \|f(t)\|_X, \quad f \in C(J_\eta : X).$$

It is known that if $-A$ is the infinitesimal generator of a C_0 semigroup $T(t)$, $t \geq 0$ and (5.1.2) has a classical solution or a strong solution u on J_T then u satisfies the integral equation

$$\begin{aligned} u(t) &= T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s,u(s)) \\ &+ \int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds, \; t \in J_T. \end{aligned} \tag{5.2.1}$$

Clearly, u given by (5.2.1) is in $C(J_T : X)$. Hence we define the mild solution to (5.1.2) as follows.

Definition 5.1 A function $u \in C(J_T : X)$ satisfying the integral equation (5.2.1) is called the **mild solution** to (5.1.2) on J_T.

We have the following result for the mild solutions to (5.1.2).

Theorem 5.1 *Let $-A$ be the infinitesimal generator of a C_0 semigroup $\{T(t), t \geq 0\}$, in X. Let $f, g : J_T \times X \to X$ be continuous in t on J_T and satisfy the following conditions.*
(H1) For every compact subinterval J_{T_0} of J_T and all $x, y \in B_R(0, X) = \{z \in X : \|z\|_X \leq R\}$ there exists a constant $L_1(T_0, R)$ such that

$$\|f(t,x) - f(t,y)\|_X \leq L_1(T_0, R)\|x - y\|_X.$$

(H2) For almost every $t \in J_T$ and all $x, y \in B_R(0, X)$ there exists a nonnegative function $L_2(., R) \in L^p_{loc}(J_T)$, $1 < p < \infty$ such that

$$\|g(t,x) - g(t,y)\|_X \leq L_1(t, R)\|x - y\|_X.$$

Further, suppose that
(H3) The real-valued map k is in $L^q(0,T)$ where $1 < q < \infty$ is such that $1/p + 1/q = 1$.
Then there exists a unique mild solution u to (5.1.2) on a compact subinterval J_{t_1} of J_T. If k is continuous on $[0, T)$, then either the mild solution u can be extended to a unique mild solution to (5.1.2) on J_T or else there exists a t_{max}, $t_0 < t_{max} < T$, such that (5.1.2) has a unique mild solution u on $[t_0, t_{max})$ and

$$\lim_{t \uparrow t_{max}} \|u(t)\| = \infty.$$

Under stronger assumptions of differentiability on the nonlinear functions f and g, we have the following regularity result establishing the existence and uniqueness of classical solution to (5.1.2).

Theorem 5.2 *Let A be the infinitesimal generator of a C_0 semigroup $\{T(t), t \geq 0\}$ on X. If $f, g : J_T \times X \to X$ are continuously differentiable from $J_T \times X$ into X and k is continuous on J_T, then the mild solution u to (5.1.2) with $u_0 \in D(A)$ is a unique classical solution to (5.1.2) on J_T.*

5.3 Proofs of Main Results

Proof of Theorem 5.1 : We fix an arbitrary $\tilde{T}$, $t_0 < \tilde{T} < T$. We first prove the existence and uniqueness of a mild solution to (5.1.2) under the assumptions that $L_1(\tilde{T}, R)$ and $L_2(t, R)$ are independent of R, i.e., $L_1(\tilde{T}, R) \equiv L_1(\tilde{T})$ and $L_2(t, R) \equiv L_2(t)$ for all $R > 0$. Let

$$\begin{aligned} M(t) &= \sup_{t \in [0,t]} \|T(t)\|_{B(X)}, \\ M_0(t) &= 1 + M(t)(L_1(\tilde{T}) + \|k\|_{L^p(J_T)} \|L_2\|_{L^q(J_T)}) \end{aligned}$$

Now, define a map $F : C(J_{\tilde{T}} : X) \to C(J_{\tilde{T}} : X)$ by

$$\begin{aligned} (Fu)(t) &= T(t - t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s, u(s)) \\ &+ \int_{t_0}^{s} k(s-\tau)g(\tau, u(\tau))\, d\tau]\, ds. \end{aligned} \tag{5.3.1}$$

Now, for $u, v \in C(J_{\tilde{T}} : X)$, we have

$$\|(Fu)(t) - (Fv)(t)\|_X \le M_0(\tilde{T})(t - t_0)\|u - v\|_{C(J_{\tilde{T}}:X)}. \tag{5.3.2}$$

Repeated applications of the above inequality gives us

$$\|(F^n u)(t) - (F^n v)(t)\|_X \le \frac{1}{n!}[M_0(\tilde{T})(t - t_0)]^n \|u - v\|_{C(J_{\tilde{T}}:X)}. \tag{5.3.3}$$

Therefore we have

$$\|F^n u - F^n v\|_{C(J_{\tilde{T}}:X)} \le \frac{1}{n!}[M_0(\tilde{T})\tilde{T}]^n \|u - v\|_{C(J_{\tilde{T}}:X)}. \tag{5.3.4}$$

From the Banach contraction principle, it follows that there is a unique fixed point $u \in C(J_{\tilde{T}} : X)$ of the map F. This u satisfies the integral equation (5.2.1) and hence it is a unique mild solution to (5.1.2) on $J_{\tilde{T}}$. Moreover, if v is the mild solution to (5.1.2) with $v(t_0) = v_0$ on $J_{\tilde{T}}$, then

$$\|u(t) - v(t)\|_X \le M_0(\tilde{T})\|u_0 - v_0\|_X + M_0(\tilde{T}) \int_{t_0}^{t} \|u - v\|_{C(J_s:X)}\, ds. \tag{5.3.5}$$

Thus, for every $\eta \in J_{\tilde{T}}$, we have

$$\|u(\eta) - v(\eta)\|_X \le M_0(\tilde{T})\|u_0 - v_0\|_X + M_0(\tilde{T}) \int_{t_0}^{\eta} \|u - v\|_{C(J_s:X)}\, ds \tag{5.3.6}$$

Taking the supremum over J_t, we get

$$\|u-v\|_{C(J_t:X)} \leq M_0(\tilde{T})\|u_0-v_0\|_X + M_0(\tilde{T})\int_{t_0}^{t}\|u-v\|_{C(J_s:X)}\, ds \quad (5.3.7)$$

Applying Gronwall's inequality and taking the supremum over $J_{\tilde{T}}$, we obtain

$$\|u-v\|_{C(J_{\tilde{T}}:X)} \leq M_0(\tilde{T})\exp\left\{M_0(\tilde{T})\tilde{T}\right\}\|u_0-v_0)\|_X. \quad (5.3.8)$$

The inequality (5.3.8) proves the uniqueness and continuous dependence of a mild solution to (5.1.2) on the initial data on $J_{\tilde{T}}$.
Now, under the general assumptions (H1) and (H2) on f and g we show that there exist a unique mild solution u to (5.1.2) on J_{t_1} for some t_1, $t_0 < t_1 < t_0+1$. Furthermore, under the assumption of continuity on k on J_T we show that either the solution can be extended on to the interval J_T or else the solution can be extended on to the maximal interval of existence $[t_0, t_{max})$, $t_0 < t_{max} < T$, and

$$\lim_{t\uparrow t_{max}}\|u(t)\|_X = \infty.$$

Let

$$\begin{aligned}
R(t_0) &= 2M(t_0+1)\|u_0\|_X,\\
N_1(t_0) &= \sup_{t_0\leq t\leq t_0+1}\|f(s,0)\|_X,\\
N_2(t_0) &= \sup_{t_0\leq t\leq t_0+1}\|g(s,0)\|_X, \qquad (5.3.9)\\
C(t_0) &= R(t_0)L_1(t_0+1,R(t_0)) + N_1(t_0)\\
&+ R(t_0)\|k\|_{L^p(J_{\tilde{T}})}\|L_2(.,R(t_0)\|_{L^q(J_{\tilde{T}})} + N_2(t_0)\|k\|_{L^p(J_{\tilde{T}})}.
\end{aligned}$$

We set

$$\delta(t_0,u_0) = \min\left\{1, \frac{\|u_0\|_X}{C(t_0)}\right\} \quad (5.3.10)$$

and let $t_1 = t_0 + \delta(t_0,u_0)$.
Now, we show that the mapping F given by (5.3.1) maps $B_{R(t_0)}(0, C(J_{t_1}:X))$ into itself. For $u \in B_{R(t_0)}(0, C(J_{t_1}:X))$, we have

$$\|F(u)(t)\|_X \leq M(t_0+1)\|u_0\|_X + M(t_0+1)\int_{t_0}^{t}[\,\|f(s,u(s)) - f(s,0)\|_X + \|f(s,0)\|_X$$

$$
\begin{aligned}
&+ \int_{t_0}^{s} |k(s-\tau)| \left[\, \|g(\tau,u(\tau)) - g(\tau,0)\|_X + \|g(\tau,0)\|_X \right] d\tau \Big] \, ds \\
&\leq M(t_0+1)[\, \|u_0\|_X + C(t_0)](t-t_0) \\
&\leq 2M(t_0+1)\|u_0\|_X \; = \; R(t_0). \qquad (5.3.11)
\end{aligned}
$$

Also, for $u, v \in B_{R(t_0)}(0, C(J_{t_1} : X))$, we have

$$\|Fu - Fv\|_{C(J_{t_1}:X)} \leq M(t_0+1)C(t_0)(t_1-t_0)\|u-v\|_{C(J_{t_1}:X)}. \qquad (5.3.12)$$

As before, the above inequality after repeated applications gives a unique fixed point u in $B_{R(t_0)}(0, C(J_{t_1} : X))$ of F as before. Now, let

$$
\begin{aligned}
r(t) &= \int_{t_0}^{t_1} k(t-\tau)g(\tau,u(\tau)) \, d\tau, \qquad (5.3.13) \\
f_1(t,u) &= f(t,u) + r(t). \qquad (5.3.14)
\end{aligned}
$$

Consider the initial value problem

$$
\begin{aligned}
\frac{dw(t)}{dt} + Aw(t) &= f_1(t,w(t)) + \int_{t_1}^{t} k(t-s)g(s,w(s)) \, ds, \quad t > t_1, \qquad (5.3.15) \\
w(t_1) &= u(t_1),
\end{aligned}
$$

Since f_1 is also continuous and satisfies (H1) with the same Lipschitz constant, therefore if $t_1 < T$ then proceeding as before, we can show that there exists a unique mild solution w to (5.3.15) on $[t_1, t_2]$, $t_2 = t_1 + \delta(t_1, \|u(t_1\|_X)$ where $\delta(t_1, \|u(t_1)\|_X)$ is defined as $\delta(t_0, \|u_0\|_X)$ by replacing t_0 and $\|u_0\|_X$ by t_1 and $\|u(t_1)\|_X$, respectively. Then the function u_1 defined on J_{t_2} in to X given by

$$u_1(t) = \begin{cases} u(t), \; t \in [t_0, t_1], \\ w(t), \; t \in [t_1, t_2] \end{cases},$$

is the mild solution to (5.1.2) on J_{t_2}. We need to check this only for $[t_1, t_2]$. If $t \in [t_1, t_2]$ then

$$
\begin{aligned}
u(t) = w(t) &= T(t-t_1)u(t_1) + \int_{t_1}^{t} T(t-s)[f_1(s,w(s)) \\
&+ \int_{t_1}^{s} k(s-\tau)g(\tau,w(\tau)) \, d\tau] \, ds
\end{aligned}
$$

$$
\begin{aligned}
&= T(t-t_1)[T(t_1-t_0)u_0 + \int_{t_0}^{t_1} T(t_1-s)[f(s,u(s)) \\
&+ \int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds] \\
&+ \int_{t_1}^{t} T(t-s)[f_1(s,w(s)) + \int_{t_1}^{s} k(s-\tau)g(\tau,w(\tau))\,d\tau]\,ds \\
&= T(t-t_0)u_0 + \int_{t_0}^{t_1} T(t-s)[f(s,u_1(s)) \\
&+ \int_{t_0}^{s} k(s-\tau)g(\tau,u_1(\tau))\,d\tau]\,ds \\
&+ \int_{t_1}^{t} T(t-s)[f(s,u_1(s)) \\
&+ r(s) + \int_{t_1}^{s} k(s-\tau)g(\tau,u_1(\tau))\,d\tau]\,ds \\
&= T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)f(s,u(s))ds \\
&+ \int_{t_0}^{t_1}\int_{t_0}^{s} T(t-s)k(s-\tau)g(\tau,u_1(\tau))\,d\tau\,ds \\
&+ \int_{t_1}^{t}\int_{t_0}^{t_1} T(t-s)k(s-\tau)g(\tau,u_1(\tau)\,d\tau\,ds \\
&+ \int_{t_1}^{t}\int_{t_1}^{s} T(t-s)k(s-\tau)g(\tau,u_1(\tau))\,d\tau\,ds \\
&= T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s,u_1(s)) \\
&+ \int_{t_0}^{s} k(s-\tau)g(\tau,u_1(\tau))\,d\tau]\,ds.
\end{aligned}
$$

Thus, either there exists a unique mild solution u to (5.1.2) on the whole of J_T or else there exists a unique mild solution on the maximal interval of existence $[t_0, t_{max})$, $t_0 < t_{max} < T$. Now, if $\lim_{t\uparrow t_{max}} \|u(t)\|_X < \infty$, then as before we can extend the mild solution beyond t_{max} which would contradict the definition of t_{max}. This completes the proof of the theorem.

The proof of Theorem 5.1 can be modified to get the following result.

Corollary 5.3 *Let A, f, g and k be as in Theorem 5.1. Let $r \in C(J_{\tilde{T}} : X)$. Then*

the integral equation

$$w(t) = r(t) + \int_{t_0}^{t} T(t-s)[f(s,w(s)) + \int_{t_0}^{s} k(s-\tau)g(\tau,w(\tau))\,d\tau]\,ds,\ t \in J_{\tilde{T}}$$

has a unique solution in $C(J_{\tilde{T}} : X)$.

Proof of Theorem 5.2 : If f and g are continuously differentiable from $J_T \times X$ into X then for any compact subinterval $J_{\tilde{T}}$ of J_T, f and g are continuous in t on $J_{\tilde{T}}$ and satisfy (H1) and (H2). Therefore (5.1.2) has a unique mild solution u on $J_{\tilde{T}}$. We shall show that u is continuously differentiable on $J_{\tilde{T}}$. Let

$$\begin{aligned}
B_1(t) &= \frac{\partial}{\partial u} f(t,u), && (5.3.16)\\
B_2(t) &= \frac{\partial}{\partial u} g(t,u), && (5.3.17)\\
r(t) &= -AT(t-t_0)u_0 + T(t-t_0)f(t_0,u_0) \\
&+ \int_{t_0}^{t} T(t-s)k(s-t_0)g(t_0,u_0)\,ds \\
&+ \int_{t_0}^{t} T(t-s)[\frac{\partial}{\partial s} f(s,u(s)) \\
&+ \int_{t_0}^{t} k(s-\tau)\frac{\partial}{\partial \tau} g(\tau,u(\tau))\,d\tau]\,ds. && (5.3.18)
\end{aligned}$$

Consider the integral equation

$$\begin{aligned}
w(t) &= r(t) + \int_{t_0}^{t} T(t-s)[B_1(s)w(s) \\
&+ \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau)\,d\tau]\,ds. && (5.3.19)
\end{aligned}$$

The assumptions on f and g imply r is continuous on $J_{\tilde{T}}$ and $b_i(t,u) = B_i(t)u$ are continuous in t from $J_{\tilde{T}}$ into X and uniformly Lipschitz continuous in u. From Corollary 5.3, it follows that (5.3.19) has a unique mild solution w on $J_{\tilde{T}}$. Now, from the assumptions on f and g we have

$$\begin{aligned}
f(s,u(s+h)) - f(s,u(s)) &= B_1(s)(u(s+h)-u(s)) \\
&+ \omega_1(s,h), && (5.3.20)
\end{aligned}$$

$$\begin{aligned}
g(\tau, u(\tau+h)) - g(\tau, u(\tau)) &= B_2(\tau)(u(\tau+h) - u(\tau)) \\
&+ \omega_2(\tau, h), \qquad (5.3.21)\\
f(s+h, u(s+h)) - f(s, u(s+h)) &= \frac{\partial}{\partial s} f(s, u(s+h)).h \\
&+ \omega_3(s, h), \qquad (5.3.22)\\
g(\tau+h, u(\tau+h)) - g(\tau, u(\tau+h)) &= \frac{\partial}{\partial \tau} g(\tau, u(\tau+h)).h \\
&+ \omega_4(s, h), \qquad (5.3.23)
\end{aligned}$$

where $h^{-1}\|w_i(s,h)\|_X \to 0$ as $h \to 0$ uniformly on $J_{\tilde{T}}$ for $i = 1, 2, 3, 4$. Let

$$w_h(t) = \frac{u(t+h) - u(t)}{h} - w(t). \qquad (5.3.24)$$

Then

$$\begin{aligned}
w_h(t) &= \left[\frac{1}{h}\left(T(t+h-t_0)u_0 - T(t-t_0)u_0\right) + AT(t-t_0)u_0\right] \\
&+ \left[\frac{1}{h}\int_{t_0}^{t_0+h} T(t+h-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds \right.\\
&- \left. T(t-t_0)f(t_0,u_0) - \int_{t_0}^{t} T(t-s)k(s-t_0)g(t_0,u_0)\,ds\right] \\
&\frac{1}{h}\left[\int_{t_0+h}^{t+h} T(t+h-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds\right.\\
&- \left.\int_{t_0}^{t} T(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds\right] \\
&- \int_{t_0}^{t} T(t-s)[\frac{\partial}{\partial s} f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)\frac{\partial}{\partial \tau} g(\tau,u(\tau))\,d\tau]\,ds \\
&- \int_{t_0}^{t} T(t-s)[B_1(s)w(s) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w(\tau)\,d\tau]\,ds. \qquad (5.3.25)
\end{aligned}$$

Now putting $s = \eta + h$ and then replacing η by s, we get

$$\begin{aligned}
&\int_{t_0+h}^{t+h} T(t+h-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds \\
&= \int_{t_0}^{t} T(t-\eta)[f(\eta+h, u(\eta+h)) \\
&+ \int_{t_0}^{\eta+h} k(\eta+h-\tau)g(\tau,u(\tau))\,d\tau]\,d\eta
\end{aligned}$$

$$= \int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) \\ + \int_{t_0}^{s+h} k(s+h-\tau)g(\tau,u(\tau))\,d\tau]\,ds \tag{5.3.26}$$

Again, in the inner integral on the right of (5.3.26) putting $\tau = \gamma + h$ and then replacing γ by τ, we get

$$\begin{aligned}
&\int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) + \int_{t_0}^{s+h} k(s+h-\tau)g(\tau,u(\tau))\,d\tau]\,ds \\
&= \int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) + \int_{t_0-h}^{s} k(s-\gamma)g(\gamma+h,u(\gamma+h))\,d\gamma]\,ds \\
&= \int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) + \int_{t_0-h}^{s} k(s-\tau)g(\tau+h,u(\tau+h))\,d\tau]\,ds \\
&= \int_{t_0}^{t} T(t-s)[f(s+h,u(s+h)) + \int_{t_0}^{s} k(s-\tau)g(\tau+h,u(\tau+h))\,d\tau]\,ds \\
&- \int_{t_0}^{t}\int_{t_0}^{t_0-h} T(t-s)k(s-\tau)g(\tau+h,u(\tau+h))\,d\tau\,ds
\end{aligned} \tag{5.3.27}$$

Now, using (5.3.27) and (5.3.20)-(5.3.23) in (5.3.25) and readjusting the terms, we get

$$\begin{aligned}
w_h(t) &= \left[\frac{1}{h}(T(t+h-t_0)u_0 - T(t-t_0)u_0) + AT(t-t_0)u_0\right] \\
&+ \left[\frac{1}{h}\int_{t_0}^{t_0+h} T(t+h-s)[f(s,u(s))\right. \\
&+ \left.\int_{t_0}^{s} k(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds - T(t-t_0)f(t_0,u_0)\right] \\
&\frac{1}{h}\int_{t_0}^{t} T(t-s)\left[\omega_1(s,h)+\omega_2(s,h) + \int_{t_0}^{s} k(s-\tau)\left\{w_3(\tau,h)+w_4(\tau,h)\right\}\,d\tau\right] ds \\
&\int_{t_0}^{t} T(t-s)\left[\left\{\frac{\partial}{\partial s}f(s,u(s+h) - \frac{\partial}{\partial s}f(s,u(s))\right\}\right. \\
&+ \left.\int_{t_0}^{s} k(s-\tau)\left\{\frac{\partial}{\partial\tau}g(\tau,u(\tau+h)) - \frac{\partial}{\partial\tau}g(\tau,u(\tau))\right\}\,d\tau\right] ds \\
&- \int_{t_0}^{t} T(t-s)\left[\frac{1}{h}\int_{t_0}^{t_0-h} g(\tau+h,u(\tau+h))\,d\tau + k(s-t_0)g(t_0,u_0)\right] ds \\
&+ \int_{t_0}^{t} T(t-s)[B_1(s)w_r(s) + \int_{t_0}^{s} k(s-\tau)B_2(\tau)w_h(\tau)\,d\tau]\,ds.
\end{aligned} \tag{5.3.28}$$

Since the norms in X of all but the term in the last line tend to zero as $h \to 0$, we have

$$\|w_h\|_{C(J_t:X)} \leq \epsilon(h) + C(\tilde{T}) \int_{t_0}^{t} \|w_h\|_{C(J_s:X)}\, ds, \tag{5.3.29}$$

where $\epsilon(h) \to 0$ as $h \to 0$ and

$$C(\tilde{T}) = \max\{\|T(t-s)\|_{B(X)}[\|B_1(s)\|_{B(X)} + \|k\|_{L^p(J)}\|B_2(s)\|_{B(X)}] : s \in J_{\tilde{T}}\}.$$

Applying Gronwall's inequality in (5.3.29), we obtain

$$\|w_h\|_{C(J_t:X)} \leq \epsilon(h) \exp\{C(\tilde{T})\tilde{T}\}. \tag{5.3.30}$$

Therefore $\|w_h(t)\|_X \to 0$ as $h \to 0$. Hence u is differentiable on $J_{\tilde{T}}$ and its derivative is w on $J_{\tilde{T}}$. Since $w \in C(J_{\tilde{T}} : X)$, $u \in C^1(J_{\tilde{T}} : X)$. Now, assumptions on f and g and $u \in C^1(J_{\tilde{T}} : X)$ imply that the maps $t \mapsto f(s, u(s))$ and $t \mapsto g(s, u(s))$ are continuously differentiable on $J_{\tilde{T}}$, it follows that

$$v(t) = T(t-t_0)u_0 + \int_{t_0}^{t} T(t-s)[f(s,u(s)) + \int_{t_0}^{s} k(s-\tau)g(\tau, u(\tau))\, d\tau]\, ds$$

is a unique classical solution to

$$\frac{dv(t)}{dt} + Av(t) = f(t,u(t)) + \int_{t_0}^{t} k(t-s)g(s,u(s))\, ds. \tag{5.3.31}$$

on $J_{\tilde{T}}$. By definition, u is a mild solution to (5.3.31) on $J_{\tilde{T}}$. By uniqueness of mild solutions to (5.3.31), we have $u = v$ on $J_{\tilde{T}}$. Thus u satisfies (5.3.31) and therefore u is a unique classical solution to (5.1.2) on $J_{\tilde{T}}$. Since $t_0 < \tilde{T} < T$, arbitrary, u is a unique classical solution to (5.1.2) on J_T. This completes the proof.

Chapter 6

Approximation of Solutions

6.1 Introduction

In this chapter we are interested in the Faedo-Galerkin approximation of solutions to the following integro-differential equation considered in a separable Hilbert space $(H, \|\ \|, (.,.))$,

$$
\begin{aligned}
u'(t) + Au(t) &= f(t, u(t)) + \int_0^t a(t-s) g(s, u(s)) ds, \quad t \geq 0, \qquad (6.1.1)\\
u(0) &= \phi.
\end{aligned}
$$

In (6.1.1), the linear operator A satisfies the following hypothesis.

(H1) A is a closed, positive definite, self-adjoint linear operator from the domain $D(A) \subset H$ into H such that $D(A)$ is dense in H, A has the pure point spectrum

$$0 < \lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$$

and a corresponding complete orthonormal system of eigenfunction $\{u_i\}$, i.e.,

$$Au_i = \lambda_i u_i \quad \text{and} \quad (u_i, u_j) = \delta_{ij},$$

where $\delta_{ij} = 1$ if $i = j$ and zero otherwise.

If **(H1)** is satisfied then $-A$ generates an analytic semigroup in H which we denote by e^{-tA}, $t \geq 0$. It follows that the fractional powers A^α of A for $0 \leq \alpha \leq 1$ are well

defined from $D(A^\alpha) \subseteq H$ into H (cf. Pazy [21], page 69-75). $D(A^\alpha)$ is a Banach space endowed with the norm

$$\|x\|_\alpha = \|A^\alpha x\|, \quad x \in D(A^\alpha).$$

On the kernel a we assume the following condition.

(H2) $a \in L^p_{loc}(0,\infty)$ for some $1 < p < \infty$. The nonlinear functions f and g are assumed to satisfy the following hypotheses.

(H3) The map f is defined from $[0,\infty) \times D(A^\alpha)$ into H and there exists a nondecreasing function F_R from $[0,\infty)$ into $[0,\infty)$ depending on $R > 0$ such that

$$\begin{aligned} \|f(t,u)\| &\leq F_R(t), \\ \|f(t,u_1) - f(t,u_2)\| &\leq F_R(t)\|u_1 - u_2\|_\alpha \end{aligned}$$

for all (t,u) and (t,u_1), (t,u_2) in $[0,\infty) \times B_R(D(A^\alpha),\phi)$, where $B_R(Z,z_0) = \{z \in Z \mid \|z - z_0\|_Z \leq R\}$ for any Banach space Z with the norm $\|\ \|_Z$.

(H4) The map g is defined from $[0,\infty) \times D(A^\alpha)$ into H and there exists a nonnegative function $G_R \in L^q_{loc}(0,\infty)$ depending on $R > 0$, where $1 < q < \infty$, $(1/p) + (1/q) = 1$, such that

$$\begin{aligned} \|g(t,u)\| &\leq G_R(t), \\ \|g(t,u_1) - g(t,u_2)\| &\leq G_R(t)\|u_1 - u_2\|_\alpha. \end{aligned}$$

for a.e. $t \in [0,\infty)$ and all u, u_1 and u_2 in $B_R(D(A^\alpha),\phi)$. Initial studies concerning existence, uniqueness and finite-time blow-up of solutions to (6.1.1) for the following special case of (6.1.1),

$$\begin{aligned} u'(t) + Au(t) &= h(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \end{aligned} \tag{6.1.2}$$

have been considered by Segal [59], Murakami [67] and Heinz and von Wahl [68]. Bazley [69, 70] has considered the following semi linear wave equation

$$\begin{aligned} u''(t) + Au(t) &= h(u(t)), \quad t \geq 0, \\ u(0) &= \phi, \\ u'(0) &= \psi, \end{aligned} \tag{6.1.3}$$

and has established the uniform convergence of approximations of solutions to (6.1.3) using the existence results of Heinz and von Wahl [68]. Goethel [71] has proved the convergence of approximations of solutions to (6.1.2) but assumed h to be defined on the whole of H. Based on the ideas of Bazley [69],[70], Miletta [72] has proved the convergence of approximations to solutions of (6.1.2). The existence, uniqueness and continuation of classical solutions to (6.1.1) is considered by Bahuguna [49]. In the present work, we use the ideas of Miletta [72] and Bahuguna [49] to establish the convergence of Faedo-Galerkin approximations of the solutions to (6.1.1) under more general conditions on the nonlinear maps f and g.

The plan of the chapter is as follows. In the second section, we consider approximate integral equations and establish the existence and uniqueness of solutions to these approximate integral equations using Banach contraction principle. In the third section, we prove the convergence of the approximate solutions to a solution to the integral equation associated with (6.1.1). We further show, in this section, that the solution can be extended to the maximal interval of existence and it is unique. In the fourth section, we consider the Faedo-Galerkin approximations and prove some results concerning the convergence of such approximations. Finally, in the fifth section, we cite some applications of the results obtained in the earlier sections.

6.2 Approximate Equations and Solutions

We shall continue to use the notations introduced in the earlier section. The existence of solutions to (6.1.1) is closely associated with the following integral equation

$$\begin{aligned} u(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f(s,u(s)) \\ &+ \int_0^s a(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds, \quad t \geq 0. \end{aligned} \tag{6.2.1}$$

In this section we shall consider an approximate integral equation to (6.2.1) and establish the existence and uniqueness of solutions to the approximate integral equation. By a solution u to (6.2.1) on $[0,T]$, $0 < T < \infty$, we mean a function $u \in X_\alpha(T)$ for some $0 < \alpha < 1$ satisfying (6.2.1) where $X_\alpha(T)$ is the Banach space

$C([0,T], D(A^\alpha))$ of all continuous functions form $[0,T]$ into $D(A^\alpha)$ endowed with the supremum norm

$$\|u\|_{X_\alpha(T)} = \sup_{0\le t\le T} \|A^\alpha u(t)\|.$$

By a solution u to (6.2.1) on $[0,\tilde{T})$, $0<\tilde{T}\le\infty$, we mean a function u such that $u\in X_\alpha(T)$ for some $0<\alpha<1$ satisfying (6.2.1) on $[0,T]$ for every $0<T<\tilde{T}$.

Since $-A$ generates the analytic semigroup e^{-tA}, $t\ge 0$, we may add cI to $-A$ for some constant c, if necessary, and assume without loss of generality that $\|e^{-tA}\|\le M$ and that $-A$ is invertible. Furthermore, it follows that A^α commutes with e^{-tA} and there exists a constant $C_\alpha>0$ depending on α such that

$$\|e^{-tA}A^\alpha\| \le C_\alpha t^{-\alpha}, \quad t>0. \tag{6.2.2}$$

Let $0<T_0<\infty$ be arbitrarily fixed and

$$L(R) = (1+R)(F_R(T_0)+\|a\|_{L^p(0,T_0)}\|G_R\|_{L^q(0,T_0)}). \tag{6.2.3}$$

Let $0<T\le T_0$ to be such that

$$T < \min\left\{T_0, \left[\frac{R}{2}(1-\alpha)(L(R)C_\alpha)^{-1}\right]^{\frac{1}{1-\alpha}}\right\}, \tag{6.2.4}$$

and

$$\sup_{0\le t\le T}\|(e^{-tA}-I)A^\alpha\phi\| \le \frac{R}{2}. \tag{6.2.5}$$

Let H_n denote the finite dimensional subspace of Hilbert space H spanned by $\{u_0,u_1,\cdots,u_n\}$ and let $P^n : H \longrightarrow H_n$ be the corresponding projection operator for $n=0,1,2,\cdots$. We define

$$f_n, g_n : [0,T]\times X_\alpha(T) \longrightarrow H,$$

$$f_n(t,u) = f(t,P^n u(t)), \quad g_n(t,u) = g(t,P^n u(t)).$$

We set $\tilde{\phi}(t)\equiv\phi$ for $t\in[0,T]$ and define a map S_n on $B_R(X_\alpha(T),\tilde{\phi})$ as follows.

$$\begin{aligned}(S_n u)(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f_n(s,u) \\ &+ \int_0^s a(s-\tau)g_n(\tau,u)\,d\tau]\,ds. \end{aligned}\tag{6.2.6}$$

Proposition 6.1 *Let* (**H1**)$-$(**H4**) *hold. Then there exists a unique* $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ *such that* $S_n u_n = u_n$ *for each* $n = 0, 1, 2, \cdots$, *i.e.,* u_n *satisfies the approximate integral equation*

$$\begin{aligned} u_n(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f_n(s,u) \\ &+ \int_0^s a(s-\tau)g_n(\tau,u)\,d\tau]\,ds. \end{aligned} \tag{6.2.7}$$

Proof: We claim that $S_n : B_R(X_\alpha(T), \tilde{\phi}) \longrightarrow B_R(X_\alpha(T), \tilde{\phi})$. For this, we need to show first that the map $t \mapsto (S_n u)(t)$ is continuous from $[0, T]$ into $D(A^\alpha)$ with respect to $\|\ \|_\alpha$ norm. For $t_1, t_2 \in [0, T]$ with $t_1 < t_2$, we have

$$\begin{aligned} \|(S_n u)(t_2) - (S_n u)(t_1)\|_\alpha &\leq \|(e^{-t_2A} - e^{-t_1A})\phi\|_\alpha \\ &+ \int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\|\ [\ \|f_n(s,u)\| \\ &+ \int_0^s |a(s-\tau)|\ \|g_n(\tau,u)\|\,d\tau]\,ds \\ &+ \int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\|\ [\ \|f_n(s,u)\| \\ &+ \int_0^s |a(s-\tau)|\ \|g_n(\tau,u)\|\,d\tau]\,ds. \end{aligned} \tag{6.2.8}$$

The first integral can be estimated as

$$\begin{aligned} \int_{t_1}^{t_2} \|e^{-(t_2-s)A}A^\alpha\|\ [\ \|f_n(s,u)\| &+ \int_0^s |a(s-\tau)|\ \|g_n(\tau,u)\|\,d\tau]\,ds \\ &\leq C_\alpha L(R)\frac{(t_2-t_1)^{1-\alpha}}{1-\alpha}. \end{aligned} \tag{6.2.9}$$

To estimate the second integral, we use part (d) of Theorem 6.13 in Pazy [21] on page 74 which states that for $0 < \beta \leq 1$ and $x \in D(A^\beta)$,

$$\|(e^{-tA} - I)x\| \leq C_\beta t^\beta \|x\|_\beta.$$

If $0 < \beta < 1$ is such that $0 < \alpha + \beta < 1$, then $A^\alpha y \in D(A^\beta)$ for any $y \in D(A^\alpha)$. Therefore, for $t, s \in [0, T]$, we have

$$\begin{aligned} \|(e^{-tA} - I)A^\alpha e^{-sA}x\| &\leq C_\beta t^\beta \|A^{(\alpha+\beta)}e^{-sA}x\| \\ &\leq C_\beta C_{\alpha+\beta} t^\beta s^{-(\alpha+\beta)}\|x\|. \end{aligned} \tag{6.2.10}$$

We use (6.2.10) to obtain the inequality

$$\begin{aligned}
&\int_0^{t_1} \|(e^{-(t_2-s)A} - e^{-(t_1-s)A})A^\alpha\| \, [\, \|f_n(s,u)\| \\
&+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u)\| \, d\tau] \, ds \\
&\le \int_0^{t_1} \|(e^{-(t_2-t_1)A} - I)e^{-(t_1-s)A}A^\alpha\| \, [\, \|f_n(s,u)\| \\
&+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u)\| \, d\tau] \, ds \\
&\le C_{\alpha,\beta}(t_2-t_1)^\beta
\end{aligned} \tag{6.2.11}$$

where

$$C_{\alpha,\beta} = C_\beta C_{\alpha+\beta} L(R) \frac{T^{1-(\alpha+\beta)}}{1-(\alpha+\beta)}.$$

From the inequalities (6.2.8), (6.2.9) and (6.2.11), it follows that $(S_n u)(t)$ is continuous from $[0,T]$ into $D(A^\alpha)$ with respect to the norm $\| \ \|_\alpha$. Now, applying A^α and subtracting $A^\alpha\phi$ from both the sides in (6.2.6), we get

$$\begin{aligned}
\|(S_n u)(t) - \phi)\|_\alpha \ \le\ & \|(e^{-tA} - I)\phi\|_\alpha + \int_0^t \|e^{-(t-s)A}A^\alpha\| \, [\, \|f_n(s,u)\| \\
&+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u)\| \, d\tau] \, ds.
\end{aligned} \tag{6.2.12}$$

It follows from (6.2.2-6.2.5) that

$$\begin{aligned}
\|S_n u(t) - \phi\|_\alpha \ &\le\ \frac{R}{2} + L(R) C_\alpha \frac{T^{1-\alpha}}{1-\alpha} \\
&\le\ R.
\end{aligned} \tag{6.2.13}$$

Taking supremum over $[0,T]$ we obtain that S_n maps $B_R(X_\alpha(T),\tilde{\phi})$ into $B_R(X_\alpha(T),\tilde{\phi})$. Now we show that S_n is a strict contraction on $B_R(X_\alpha(T),\tilde{\phi})$. For $u,v \in B_R(X_\alpha(T),\tilde{\phi})$, we have

$$\begin{aligned}
\|(S_n u)(t) - (S_n v)(t))\|_\alpha \ \le\ & \int_0^t \|e^{-(t-s)A}A^\alpha\| \, [\, \|f_n(s,u) - f_n(s,v)\| \\
&+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u) - g_n(\tau,v)\| \, d\tau] \, ds \\
\le\ & (F_R(T_0) + \|a\|_{L^p(0,T_0)} \|G_R\|_{L^q(0,T_0)})
\end{aligned}$$

$$
\begin{aligned}
C_\alpha \frac{T^{1-\alpha}}{1-\alpha}\|u-v\|_{X_\alpha(T)} &\leq \frac{1}{R}(RF_R(T_0) + R\|a\|_{L^p(0,T_0)}\|G_R\|_{L^q(0,T_0)}) \\
C_\alpha \frac{T^{1-\alpha}}{1-\alpha}\|u-v\|_{X_\alpha(T)} &\leq \frac{1}{R}L(R)C_\alpha \frac{T^{1-\alpha}}{1-\alpha}\|u-v\|_{X_\alpha(T)} \\
&\leq \frac{1}{2}\|u-v\|_{X_\alpha(T)}. \qquad (6.2.14)
\end{aligned}
$$

Here we have used (6.2.4) to obtain the last inequality. Taking supremum over $[0, T]$, we get

$$\|S_n u - S_n v\|_{X_\alpha(T)} \leq \frac{1}{2}\|u - v\|_{X_\alpha(T)}.$$

Thus $S_n u$ is a strict contraction on $B_R(X_\alpha(T), \tilde{\phi})$. Therefore, there exists a unique $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ such that $S_n u_n = u_n$. Clearly, u_n satisfies (6.2.7). This completes the proof of the proposition.

Proposition 6.2 *Let* **(H1)** $-$ **(H4)** *hold. If* $\phi \in D(A^\alpha)$ *for some* $0 < \alpha < 1$, *then* $u_n(t) \in D(A^\beta)$ *for all* $t \in (0, T]$ *where* $0 \leq \beta < 1$. *Furthermore, if* $\phi \in D(A)$ *then* $u_n(t) \in D(A^\beta)$ *for all* $t \in [0, T]$ *where* $0 \leq \beta < 1$.

Proof: From the Proposition 6.1 we have the existence of a unique $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ satisfying (6.2.7). Part (a) of Theorem 6.13 in Pazy [21] on page 74 implies that $e^{-tA} : H \longrightarrow D(A^\beta)$ for $t > 0$ and $0 \leq \beta < 1$. Also, from Theorem 2.4 in Pazy [21] on page 4, we have $e^{-tA}x \in D(A)$ if $x \in D(A)$. The results of the proposition follow form these facts and the fact that $D(A) \subseteq D(A^\beta)$ for $0 \leq \beta \leq 1$.

Proposition 6.3 *Let* **(H1)** $-$ **(H4)** *hold. Then, for any* $\phi \in D(A^\alpha)$, $0 < \alpha < 1$ *and any* $t_0 \in (0, T]$ *there exists a constant* U_{t_0}, *independent of* n, *such that*

$$\|u_n(t)\|_\beta \leq U_{t_0}, \quad 0 \leq \beta < 1, \quad t_0 \leq t \leq T.$$

Moreover, if $\phi \in D(A)$, *then there exists a constant* U_0, *independent on* n, *such that*

$$\|u_n(t)\|_\beta \leq U_0, \quad 0 \leq \beta < 1, \quad 0 \leq t \leq T.$$

Proof: Applying A^β on both the sides in (6.2.7) and using part (c) of Theorem 6.13 in Pazy [21] on page 74 after taking the norm, we have, for $t_0 \leq t \leq T$,

$$\|u_n(t)\|_\beta \leq C_\beta t_0^{-\beta}\|\phi\| + C_\beta L(R)\frac{T^{1-\beta}}{1-\beta} \leq U_{t_0}. \qquad (6.2.15)$$

If $\phi \in D(A)$, then $\phi \in D(A^\beta)$ for $0 \le \beta \le 1$ and we get

$$\|u_n(t)\|_\beta \;\le\; M\|A^\beta\phi\| + C_\beta L(R)\frac{T^{1-\beta}}{1-\beta} \le U_0. \tag{6.2.16}$$

This completes the proof of the proposition.

6.3 Convergence

In this section we prove the convergence of the solution $u_n \in X_\alpha(T)$ of the approximate integral equation

$$\begin{aligned} u_n(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f_n(s,u_n) \\ &+ \int_0^s a(s-\tau)g_n(\tau,u_n)\,d\tau]\,ds \end{aligned} \tag{6.3.1}$$

to a unique solution u of (6.2.1).

Proposition 6.4 *Let* **(H1)** – **(H4)** *hold. If* $\phi \in D(A^\alpha)$, $0 < \alpha < 1$, *then for any* $t_0 \in (0,T]$

$$\lim_{m\to 0} \sup_{\{n\ge m,\ t_0\le t\le T\}} \|u_n(t)-u_m(t)\|_\alpha \;=\; 0.$$

Proof: For $n \ge m$, we have

$$\begin{aligned} \|f_n(t,u_n)-f_m(t,u_m)\| &\le \|f_n(t,u_n)-f_n(t,u_m)\| \\ &+ \|f_n(t,u_m)-f_m(t,u_m)\| \\ &\le F_R(T_0)[\,\|u_n(t)-u_m(t)\|_\alpha \\ &+ \|(P^n-P^m)u_m(t)\|_\alpha]. \end{aligned} \tag{6.3.2}$$

For $0<\alpha<\beta<1$, we have

$$\begin{aligned} \|A^\alpha(P^n-P^m)u_m(t)\| &\le \|A^{\alpha-\beta}(P^n-P^m)u_m(t)A^\beta\| \\ &\le \frac{1}{\lambda_m^{\beta-\alpha}}\|A^\beta u_m(t)\|. \end{aligned} \tag{6.3.3}$$

Using (6.3.3) in (6.3.2), we get

$$\begin{aligned} \|f_n(t,u_n)-f_m(t,u_m)\| &\le F_R(T_0)[\,\|u_n(t)-u_m(t)\|_\alpha \\ &+ \frac{1}{\lambda_m^{\beta-\alpha}}\|u_m(t)\|_\beta]. \end{aligned} \tag{6.3.4}$$

Similarly, we have

$$\begin{aligned} \|g_n(t,u_n) - g_m(t,u_m)\| &\leq G_R(t)[\, \|u_n(t) - u_m(t)\|_\alpha \\ &+ \frac{1}{\lambda_m^{\beta-\alpha}}\|u_m(t)\|_\beta]. \end{aligned} \tag{6.3.5}$$

Now, for $0 < t_0' < t_0$, we have

$$\begin{aligned} \|u_n(t) - u_m(t)\|_\alpha &\leq \left(\int_0^{t_0'} + \int_{t_0'}^t\right) \|e^{-(t-s)A}A^\alpha\| \, [\, \|f_n(s,u_n) - f_n(s,u_m)\| \\ &+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u_n) - g_m(\tau,u_m)\| \, d\tau] \, ds. \end{aligned} \tag{6.3.6}$$

The first integral can be estimated as

$$\begin{aligned} &\int_0^{t_0'} \|e^{-(t-s)A}A^\alpha\| \, [\, \|f_n(t,u_n) - f_n(t,u_m)\| \\ &+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u_n) - g_m(\tau,u_m)\| \, d\tau] \, ds \\ &\leq 2L(R)C_\alpha(t_0 - t_0')^{-\alpha}t_0'. \end{aligned} \tag{6.3.7}$$

For the second integral, we have

$$\begin{aligned} &\int_{t_0'}^t \|e^{-(t-s)A}A^\alpha\| \, [\, \|f_n(t,u_n) - f_n(t,u_m)\| \\ &+ \int_0^s |a(s-\tau)| \, \|g_n(\tau,u_n) - g_m(\tau,u_m)\| \, d\tau] \, ds \\ &\leq L(R)C_\alpha \frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \frac{T^{1-\alpha}}{1-\alpha} \\ &+ L(R)C_\alpha \int_{t_0'}^t \frac{1}{(t-s)^\alpha}\|u_n - u_m\|_{X_\alpha(s)} \, ds. \end{aligned} \tag{6.3.8}$$

Using the inequalities (6.3.7) and (6.3.8) in (6.3.6), we obtain

$$\begin{aligned} \|u_n - u_m\|_{X_\alpha(t)} &\leq 2L(R)C_\alpha(t_0 - t_0')^{-\alpha}t_0' \\ &+ L(R)C_\alpha \frac{U_{t_0'}}{\lambda_m^{\beta-\alpha}} \frac{T^{1-\alpha}}{1-\alpha} \\ &+ L(R)C_\alpha \int_{t_0'}^t \frac{1}{(t-s)^\alpha}\|u_n - u_m\|_{X_\alpha(s)} \, ds. \end{aligned} \tag{6.3.9}$$

Applying Gronwall's inequality in (6.3.9) then letting $m \to \infty$ and using the fact that t_0' is arbitrary, we get the required result. This completes the proof of the proposition. In view of Propositions 6.2 and 6.3, we have the following result.

Corollary 6.1 *If $\phi \in D(A)$, then*

$$\lim_{m\to 0} \sup_{\{n\geq m,\, 0\leq t\leq T\}} \|u_n(t) - u_m(t)\|_\alpha = 0.$$

For the convergence of the solution $u_n(t)$ of the approximate equation (6.3.1) we have the following result.

Theorem 6.2 *Let* **(H1)** − **(H4)** *hold and let $\phi \in D(A^\alpha)$. Then there exists a function $u \in X_\alpha(T)$ such that $u_n \to u$ as $n \to \infty$ in $X^\alpha(T)$ and u satisfies* (6.2.1) *on* $[0,T]$. *Furthermore, u can be extended to the maximal interval of existence $[0, t_{max})$, $0 < t_{max} \leq \infty$ satisfying* (6.2.1) *on $[0, t_{max})$ and u is a unique solution to* (6.2.1) *on* $[0, t_{max})$.

Proof: We first assume that $\phi \in D(A)$. Corollary 6.1 implies that there exists $u \in X_\alpha(T)$ such that u_n converges to u in $X_\alpha(T)$. Since $u_n \in B_R(X_\alpha(T), \tilde{\phi})$ for each n, u is also in $B_R(X^\alpha(T), \tilde{\phi})$. Further, we have

$$\begin{aligned} \|f_n(t,u_n) - f(t,u(t))\| &= \|f(t, P^n u_n(t)) - f(t,u(t))\| \\ &\leq F_R(T_0)[\,\|u_n(t) - u(t)\|_\alpha \\ &+ \|(P^n - I)u(t)\|_\alpha]. \end{aligned} \tag{6.3.10}$$

Taking supremum over $[0,T]$, we get

$$\begin{aligned} \sup_{0\leq t\leq T} \|f_n(t,u_n) &- f(t,u(t))\| \\ &\leq F_R(T_0)[\,\|u_n - u\|_{X_\alpha(T)} \\ &+ \|P^n - I\|\,\|u\|_{X_\alpha(T)}] \to 0 \quad \text{as} \quad n\to\infty. \end{aligned} \tag{6.3.11}$$

Similarly, for g we have

$$\begin{aligned} \|g_n(t,u_n) - g(t,u(t))\| &= \|g(t, P^n u_n(t)) - g(t,u(t))\| \\ &\leq G_R(t)[\,\|u_n(t) - u(t)\|_\alpha \\ &+ \|(P^n - I)u(t)\|_\alpha]. \end{aligned} \tag{6.3.12}$$

From (6.3.12), we have

$$\begin{aligned}&\sup_{0\le t\le T}\int_0^t |a(t-s)|\,\|g_n(s,u_n)-g(s,u(s))\|\,ds\\ &\le L(R)[\,\|u_n-u\|_{X_\alpha(T)}\\ &+\|P^n-I\|u\|_{X_\alpha(T)}]\to 0 \quad \text{as} \quad n\to\infty.\end{aligned} \tag{6.3.13}$$

Using (6.3.10) and (6.3.12) in (6.3.1), we get

$$\begin{aligned}u(t) &= e^{-tA}\phi+\int_0^t e^{-(t-s)A}[f(s,u(s))\\ &+\int_0^s a(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds.\end{aligned} \tag{6.3.14}$$

Now, let $\phi\in D(A^\alpha)$. Since for $0<t\le T$, $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ and $u_n(0)=u(0)=\phi$, we have, for $0\le t\le T$, $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ in H. Furthermore, since each u_n is in $B_R(X_\alpha(T),\tilde\phi)$ we have $u\in B_R(X_\alpha(T),\tilde\phi)$ and for any $0<t_0\le T$,

$$\lim_{n\to\infty}\sup_{t_0\le t\le T}\|u_n(t)-u(t)\|_\alpha = 0. \tag{6.3.15}$$

Also,

$$\begin{aligned}\sup_{t_0\le t\le T}\|f_n(t,u_n) &- f(t,u(t))\|\\ &\le F_R(T_0)[\,\|u_n-u\|_{X_\alpha(T)}\\ &+ \|P^n-I\|\,\|u\|_{X_\alpha(T)}]\to 0 \quad \text{as} \quad n\to\infty\end{aligned} \tag{6.3.16}$$

and

$$\begin{aligned}\sup_{t_0\le t\le T}\int_0^t |a(t-s)|\,\|g_n(s,u_n) &- g(s,u(s))\|\,ds\\ &\le L(R)[\,\|u_n-u\|_{X_\alpha(T)}\\ &+ \|P^n-I\|u\|_{X_\alpha(T)}]\to 0 \quad \text{as} \quad n\to\infty\end{aligned} \tag{6.3.17}$$

Now, for $0<t_0<t$, we may rewrite (6.3.1) as

$$\begin{aligned}u_n(t) &= e^{-tA}\phi+\left(\int_0^{t_0}+\int_{t_0}^t\right)e^{-(t-s)A}[f_n(s,u_n)\\ &+\int_0^s a(s-\tau)g_n(\tau,u_n)\,d\tau]\,ds.\end{aligned} \tag{6.3.18}$$

For the first integral, we have

$$\begin{aligned}\|\int_0^{t_0} e^{-(t-s)A}[f_n(s,u_n) &+ \int_0^s a(s-\tau)g_n(\tau,u_n)\,d\tau]\,ds\| \\ &\le ML(R)t_0. \qquad (6.3.19)\end{aligned}$$

Thus, we have

$$\begin{aligned}\|u_n(t) - e^{-tA}\phi &- \int_{t_0}^{t} e^{-(t-s)A}[f_n(s,u_n) \\ &+ \int_0^s a(s-\tau)g_n(\tau,u_n)\,d\tau]\,ds\| \\ &\le ML(R)t_0. \qquad (6.3.20)\end{aligned}$$

Letting $n \to \infty$ in (6.3.20), we get

$$\begin{aligned}\|u(t) - e^{-tA}\phi &- \int_{t_0}^{t} e^{-(t-s)A}[f(s,u(s)) \\ &+ \int_0^s a(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds\| \\ &\le ML(R)t_0. \qquad (6.3.21)\end{aligned}$$

Since $0 < t_0 \le T$ is arbitrary, we obtain that u satisfies the integral equation (6.2.1). If u satisfies (6.2.1) on $[0,T]$, then we show that u can be extended further. Since $0 < T_0 < \infty$ was arbitrary, we assume that $0 < T < T_0$. We consider the equation

$$\begin{aligned}\frac{dw(t)}{dt} + Aw(t) &= F(t,w(t)) \\ &+ \int_0^t a(t-s)G(s,w(s))\,ds, \quad 0 \le t \le T_0 < \infty, \qquad (6.3.22) \\ w(0) &= u(T),\end{aligned}$$

where $F, G : [0, T_0 - T] \times D(A^\alpha) \longrightarrow H$ are defined by

$$\begin{aligned}F(t,x) &= f(t+T,x) + h(t), \\ h(t) &= \int_0^T a(t+T-s)g(s,u(s))\,ds, \\ G(t,x) &= g(t+T,x),\end{aligned}$$

for $(t,x) \in [0, T_0 - T] \times D(A^\alpha)$. We note that F and G satisfy **(H3)** and **(H4)**, respectively, for T_0 replaced by $T_0 - T$. Hence there exists a $w \in C([0,T_1], D(A^\alpha))$ for some $0 < T_1 \le T_0 - T$ satisfying the integral equation

$$\begin{aligned} w(t) &= e^{-tA}u(T) + \int_0^t e^{-(t-s)A}[F(s,w(s)) \\ &+ \int_0^s a(s-\tau)G(\tau,w(\tau))\,d\tau]\,ds, \quad 0 \le t \le T_1. \end{aligned} \tag{6.3.23}$$

We define

$$\tilde{u}(t) = \begin{cases} u(t), & 0 \le t \le T, \\ w(t-T), & T \le t \le T_1 + T. \end{cases}$$

Then $\tilde{u}$ satisfies the integral equation

$$\begin{aligned} \tilde{u}(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f(s,\tilde{u}(s)) \\ &+ \int_0^s a(s-\tau)g(\tau,\tilde{u}(\tau))\,d\tau]\,ds, \quad 0 \le t \le T_1 + T. \end{aligned} \tag{6.3.24}$$

To see this, we need to verify (6.3.24) only on $[T, T_1 + T]$. For $t \in [T, T_1 + T]$,

$$\begin{aligned} \tilde{u}(t) &= w(t-T) \\ &= e^{-(t-T)A}u(T) + \int_0^{t-T} e^{-(t-T-s)A}[F(s,w(s)) \\ &+ \int_0^s a(s-\tau)G(\tau,w(\tau))\,d\tau]\,ds \\ &= e^{-(t-T)A}\Big[e^{-TA}\phi + \int_0^T e^{-(T-s)A}[f(s,u(s)) \\ &+ \int_0^s a(s-\tau)g(\tau,u(\tau))\,d\tau]\,ds\Big] + \int_0^{t-T} e^{-(t-T-s)A}[F(s,w(s)) \\ &+ \int_0^s a(s-\tau)G(\tau,w(\tau))\,d\tau]\,ds. \end{aligned} \tag{6.3.25}$$

Putting $s + T = \eta$ and $\tau + T = \xi$, we have

$$\begin{aligned} \tilde{u}(t) &= e^{-tA}\phi + \int_0^T e^{-(t-s)A}[f(s,\tilde{u}(s)) \\ &+ \int_0^s a(s-\tau)g(\tau,\tilde{u}(\tau))\,d\tau]\,ds \end{aligned}$$

$$
\begin{aligned}
& + \int_T^t e^{-(t-\eta)A}[f(\eta,\tilde{u}(\eta)) + h(\eta - T) \\
& + \int_T^\eta a(\eta-\xi)g(\xi,\tilde{u}(\xi))\,d\xi]\,d\eta \\
&= e^{-tA}\phi + \int_0^t e^{-(t-s)A} f(s,\tilde{u}(s))\,ds \\
& + \int_0^T \int_0^s e^{-(t-s)A} a(s-\tau)g(\tau,\tilde{u}(\tau))\,d\tau\,ds \\
& + \int_T^t \int_0^T e^{-(t-s)A} a(s-\tau)g(\tau,\tilde{u}(\tau))\,d\tau\,ds \\
& + \int_T^t \int_T^s e^{-(t-s)A} a(s-\tau)g(\tau,\tilde{u}(\tau))\,d\tau\,ds \\
&= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f(s,\tilde{u}(s)) \\
& + \int_0^s a(s-\tau)G(\tau,\tilde{u}(\tau))\,d\tau]\,ds. \qquad (6.3.26)
\end{aligned}
$$

Thus, $\tilde{u}(t)$ satisfies (6.2.1) on $[0, T_1+T]$. Hence, we may extend $u(t)$ to the maximal interval $[0, t_{max})$ satisfying (6.2.1) on $[0, t_{max})$, $0 < t_{max} \leq \infty$.

Now we show the uniqueness. Let u_1 and u_2 be two solutions of (6.2.1). Let T any number such that $0 < T < t_{max}$. Let

$$R = \max\{\|u_1 - \tilde{\phi}\|_{X_\alpha(T)}, \|u_2 - \tilde{\phi}\|_{X_\alpha(T)}\}.$$

Then, for $0 < \eta \leq T$, we have

$$
\begin{aligned}
\|u_1(\eta) - u_2(\eta)\|_\alpha &\leq \int_0^\eta \|e^{-(\eta-s)A}A^\alpha\|\,\|\,[f(s,u_1(s)) - f(s,u_2(s))\| \\
&+ \int_0^s |a(s-\tau)|\,\|g(\tau,u_1(\tau)) - g(\tau,u_2(\tau))\|\,d\tau]\,ds \\
&\leq L(R)C_\alpha \int_0^\eta (\eta-s)^{-\alpha}\|u_1 - u_2\|_{X_\alpha(s)}\,ds. \qquad (6.3.27)
\end{aligned}
$$

Taking supremum on $0 \leq \eta \leq t$ in (6.3.27), we obtain

$$\|u_1 - u_2\|_{X_\alpha(t)} \leq L(R)C_\alpha \int_0^\eta (\eta-s)^{-\alpha}\|u_1 - u_2\|_{X_\alpha(s)}\,ds. \qquad (6.3.28)$$

From Gronwall's inequality and the fact that

$$\|u_1(t) - u_2(t)\| \leq \frac{1}{\lambda_0^\alpha}\|u_1 - u_2\|_{X_\alpha(T)},$$

it follows that $u_1 = u_2$ on $[0, T]$. Since $0 < T < t_{max}$ was arbitrary, we have $u_1 = u_2$ on $[0, t_{max})$. This completes the proof of the theorem.

Corollary 6.3 *If there exists a continuous function $K : [0, \infty) \longrightarrow [0, \infty)$ such that*

$$\|u(t)\|_\alpha \le K(t), \quad t \ge 0,$$

then u satisfies (6.2.1) on $[0, \infty)$.

6.4 Faedo-Galerkin Approximations

For any $0 < T < t_{max}$, we have a unique $u \in X^\alpha(T)$ satisfying the integral equation

$$\begin{aligned} u(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f(s, u(s)) \\ &+ \int_0^s a(s-\tau)g(\tau, u(\tau))\, d\tau]\, ds. \end{aligned} \tag{6.4.1}$$

Also, we have a unique solution $u_n \in X^\alpha(T)$ of the approximate integral equation

$$\begin{aligned} u_n(t) &= e^{-tA}\phi + \int_0^t e^{-(t-s)A}[f(s, P^n u_n(s)) \\ &+ \int_0^s a(s-\tau)g(\tau, P^n u_n(\tau))\, d\tau]\, ds. \end{aligned} \tag{6.4.2}$$

If we project equation (6.4.2) onto H_n, we get the Faedo-Galerkin approximation $\hat{u}_n(t) = P^n u_n(t)$ satisfying

$$\begin{aligned} \hat{u}(t) &= e^{-tA}P^n\phi + \int_0^t e^{-(t-s)A}P^n[f(s, \hat{u}_n(s)) \\ &+ \int_0^s a(s-\tau)g(\tau, \hat{u}_n(\tau))\, d\tau]\, ds. \end{aligned} \tag{6.4.3}$$

The solutions u of (6.4.1) and $\hat{u}_n$ of (6.4.3), have the representations

$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)u_i, \quad \alpha_i(t) = (u(t), u_i) \quad i = 0, 1, \cdots \tag{6.4.4}$$

$$\hat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)u_i, \quad \alpha_i(t) = (\hat{u}_n(t), u_i) \quad i = 0, 1, \cdots \tag{6.4.5}$$

Using (6.4.5) in (6.4.2), we obtain a system of first order integrodifferential equations

$$\begin{aligned} \frac{d\alpha_i^n(t)}{dt} + \lambda_i \alpha_i^n(t) &= F_i^n(t, \alpha_0^n(t), \cdots, \alpha_n^n(t)) \\ &+ \int_0^t a(t-s) G_i^n(t, \alpha_0^n(s), \cdots, \alpha_n^n(s))\, ds \qquad (6.4.6) \\ \alpha_i^n(0) &= \phi_i, \qquad (6.4.7) \end{aligned}$$

where

$$\begin{aligned} F_i^n(t, \alpha_0^n(t), \cdots, \alpha_n^n(t)) &= \left(f(t, \sum_{i=0}^n \alpha_i^n(t) u_i), u_i \right), \\ G_i^n(t, \alpha_0^n(t), \cdots, \alpha_n^n(t)) &= \left(g(t, \sum_{i=0}^n \alpha_i^n(t) u_i), u_i \right) \end{aligned}$$

and $\phi_i = (\phi, u_i)$ for $i = 1, 2, \cdots, n$. For the convergence of α_i^n to α_i, we have the following convergence theorem.

Theorem 6.4 *Let* (**H1**) – (**H4**) *hold. Then we have the following.*
(a) If $\phi \in D(A^\alpha)$, then for any $0 < t_0 \leq T$,

$$\lim_{n\to\infty} \sup_{t_0 \leq t \leq T} \left[\sum_{i=0}^n \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2 \right] = 0. \qquad (6.4.8)$$

(b) If $\phi \in D(A)$, then

$$\lim_{n\to\infty} \sup_{0 \leq t \leq T} \left[\sum_{i=0}^n \lambda_i^{2\alpha} \{\alpha_i(t) - \alpha_i^n(t)\}^2 \right] = 0. \qquad (6.4.9)$$

Proof:

$$\begin{aligned} A^\alpha[u(t) - \hat{u}_n(t)] &= A^\alpha \left[\sum_{i=0}^\infty \{\alpha_i(t) - \alpha_i^n(t)\} u_i \right] \\ &= \sum_{i=0}^\infty \lambda_i^\alpha \{\alpha_i(t) - \alpha_i^n(t)\} u_i. \end{aligned}$$

Thus

$$\|A^\alpha[u(t) - \hat{u}_n(t)]\|^2 \geq \sum_{i=0}^n \lambda_i^{2\alpha} [\alpha_i(t) - \alpha_i^n(t)]^2. \qquad (6.4.10)$$

To conclude the results of the theorem, we may use the following result a proof of which is similar to the proofs of Proposition 6.4 and Corollary 6.1.

Proposition 6.5 *Let* (H1) – (H4) *hold and let* T *be any number such that* $0 < T < t_{max}$. *Then we have the following.*
(a) If $\phi \in D(A^\alpha)$, *then for any* $0 < t_0 < T$,

$$\lim_{n\to\infty} \sup_{\{n\geq m,\ t_0\leq t\leq T\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

(b) If $\phi \in D(A)$, *then*

$$\lim_{n\to\infty} \sup_{\{n\geq m,\ 0\leq t\leq T\}} \|A^\alpha[\hat{u}_n(t) - \hat{u}_m(t)]\| = 0.$$

6.5 Applications

We consider the following class of integrodifferential equations

$$\begin{aligned} u_t - \Delta u &= f_1(t,u,\nabla u) \\ &+ \int_0^t a(t-s) f_2(s,u(s),\nabla u(s))\, ds, \quad x \in \Omega, \quad t > 0, \qquad (6.5.1) \\ u(t,x) &= 0, \quad x \in \partial\Omega, \\ u(0,x) &= u_0(x) \end{aligned}$$

Here $\Omega \subset \mathbf{R}^3$ is a bounded domain with sufficiently smooth boundary $\partial\Omega$ and Δ, ∇ are the 3-dimensional Laplacian and gradient, respectively. We assume that $a \in L^p_{loc}(0,\infty)$ and $f_i(t,u,p)$, $(t,u,p) \in \mathbf{R}_+ \times \mathbf{R} \times \mathbf{R}^3$, $i = 1,2$, are locally Lipschitz continuous functions of all its arguments. We further assume that there are functions $\rho_i : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ and a real constant $\gamma, 1 \leq \gamma < 3$ such that

$$\begin{aligned} |f_i(t,u,p)| &\leq \rho_i(t,|u|)(1+|p|^\gamma), \\ |f_i(t,u,p) - f_i(t,u,q)| &\leq \rho_i(t,|u|)(1+|p|^{\gamma-1}+|q|^{\gamma-1})|p-q|, \\ |f_i(t,u,p) - f_i(t,v,p)| &\leq \rho_i(t,|u|+|v|)(1+|p|^\gamma)|u-v| \end{aligned}$$

for all t for $i = 1$ and a.e. t for $i = 2$ where $\rho_1(.,r)$ is nondecreasing and $\rho_2(.,r) \in L^q_{loc}(0,\infty)$ for each $r \geq 0$ where $(1/p) + (1/q) = 1$.

We reformulate (6.5.1) as an abstract integrodifferential equation (6.1.1) in the

Hilbert space $\mathbf{H} = L^2(\Omega)$ where $A = -\Delta + cI$, for some $c > 0$ with $D(A) = \mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega)$, $\mathbf{H}^2(\Omega)$, $\mathbf{H}_0^1(\Omega)$ are Sobolev spaces (cf. §7.1 in [21] for the definitions) and I is the identity operator.

We define the maps $f, g : [0, \infty) \times D(A^\alpha)$ by

$$\begin{aligned} f(t,u)(x) &= f_1(t, u(x), \nabla u(x)) + cu(x), \\ g(t,u)(x) &= f_2(t, u(x), \nabla u(x)), \end{aligned}$$

where $\max\{3/4, (5\gamma - 3)/4\gamma\} < \alpha < 1$ and $\phi \in \mathbf{H}$ is given by $\phi(x) = u_0(x)$. We note **(H1)**-**(H4)** are satisfied (for **(H3)** and **(H4)** cf. Theorem 4.4 pp. 244-245 in Pazy [21]). Thus, we may apply the results of the earlier sections to guarantee the existence of Faedo-Galerkin approximations and their convergence to the unique solution of (6.5.1).

Future Scope

In this chapter we mention some of the problems which we plan to study in the near future.

1. **Evolution Systems and Applications**

   All the problems we have considered in the present work, the operators appearing in the abstract Cauchy problems does not depend on time. In many practical applications, the coefficients of the differential operators are in general depending on time. Therefore in the abstract formulations of such problems, the associated operators will also be time dependent. We plan to study such problems with the help of some modifications in the ideas and techniques developed in the present work.

2. **Quasi-linear Evolution Equations**

   More generally, as mentioned above, the coefficients of the partial differential operators will depend on the unknown solution of the problem. Therefore the associated operator in the abstract Cauchy problem will involve the unknown function. Many of such problems are in the domain of quasi-linear evolution equations. We plan to modify the techniques of the method of semi-discretization in time to study these quasi-linear evolution equations.

3. **Approximation Results for Hyperbolic Problems**

   In chapter Six, we have considered the Fadeo-Galerkin approximation of solutions to an integrodifferential equation. This integrodifferential equation is an abstract formulation of a parabolic integrodifferential equation and therefore the assumption that the associated operator is the infinitesimal generator of

analytic semigroup. As mentioned earlier, this allows us to consider the fractional powers of the operator. For a hyperbolic problem, if we consider its abstract formulation as a first order equation in a Banach space, the associated operator is the infinitesimal generator of a strongly continuous semigroup only. This restricts the analysis of such problems considerably. We plan to consider the abstract formulations of such problems as second order equation and the associated operator will still be generating analytic semigroup. We hope to modify our techniques suitably in order to study such problems and prove some approximation results.

List of Papers Accepted and Communicated

1. D. Bahuguna, S. K. Srivastava and S. Singh, " Approximations of solutions to semilinear integrodifferential equation ", *Numerical Functional Analysis and Optimization* , Accepted.

2. D. Bahuguna, S. Singh and R.K. Shukla, "Application of Method of Semidiscritization in time to Linear Viscoelastic System With Small Strains", *J. Differential Equation and Dynamical Systems*, Communicated.

3. D. Bahuguna, S.Singh, " Regularity of Solution to Semilinear Hyperbolic Integrodifferential Equations", Communicated.

4. D.Bahuguna, S. Singh, and R.K. Shukla, "Application of Semigroup Theory to Linear Viscoelastic Systems with Finite Strains", *J. Differential Equation and Dynamical Systems*, Communicated.

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